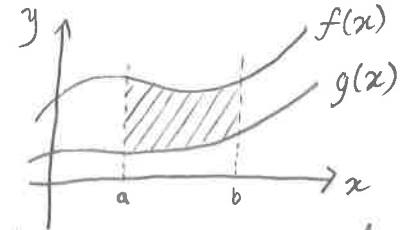
7.2 Areas in the Plane

Definition: Area between curves

f, g continuous, $f(x) \geq g(x)$ on $[a,b] \Rightarrow$ area between $f(x)$ and $g(x)$ on $[a,b]$ is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

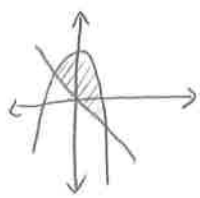


Ex. 1) Find the area between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.

$\sec^2 x \geq \sin x$ on $[0, \pi/4] \Rightarrow A = \int_0^{\pi/4} [\sec^2 x - \sin x]\, dx = \tan x + \cos x \Big|_0^{\pi/4}$

$= \left(1 + \frac{1}{\sqrt{2}}\right) - (0 + 1) = \frac{1}{\sqrt{2}}$

Ex. 2) Find the area between $y = 2 - x^2$ and $y = -x$.



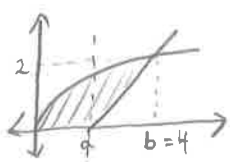
$a = ?,\ b = ?$ $\quad 2 - x^2 = -x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0$

$\Rightarrow x = -1$ or $x = 2 \Rightarrow a = -1$ & $b = 2$

$A = \int_{-1}^{2} (2 - x^2) - (-x)\, dx = \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$

$= \left(4 - \frac{8}{3} + \frac{4}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) = \left(10/3\right) - \left(-7/6\right) = 9/2$

Ex. 3) Find the area between $y = \sqrt{x}$, $y = x - 2$, and $y = 0$.



$a = ?,\ b = ?$ $\quad \sqrt{x} = x - 2 \Rightarrow x = x^2 - 4x + 4 \Rightarrow x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0 \Rightarrow x = 4$ or $x = 1 \Rightarrow b = 4$

$0 = x - 2 \Rightarrow x = 2 \Rightarrow a = 2$

$A = \int_0^2 \sqrt{x}\, dx + \int_2^4 \sqrt{x} - (x-2)\, dx = \frac{x^{3/2}}{3/2}\Big|_0^2 + \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right]_2^4$

$= \frac{2}{3} 2^{3/2} + \left(\frac{2}{3} 4^{3/2} - \cancel{8} + \cancel{8}\right) - \left(\frac{2}{3} 2^{3/2} - 2 + 4\right) = \frac{16}{3} - 2 = \frac{10}{3}$

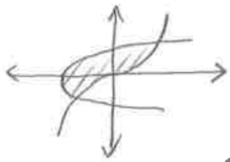
We can also integrate with respect to y to find the area.

Ex. 4) Redo example 3 integrating with respect to y.

$y = \sqrt{x} \Rightarrow y^2 = x$, $\quad y = x - 2 \Rightarrow x = y + 2$, when $x = 4$ $y = \sqrt{x} = \sqrt{4} = 2$

$A = \int_0^2 (y+2) - y^2\, dy = \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_0^2 = 2 + 4 - \frac{8}{3} = \frac{10}{3}$

Ex. 5) Find the area bound by the curves $y = x^3$ and $x = y^2 - 2$.



Integration on y is easier. Find limits $x = (x^3)^2 - 2$
$\Rightarrow x = x^6 - 2 \Rightarrow x^6 - x - 2 = 0 \Rightarrow x = 1.215$ or $x = -1$ using a calculator $\Rightarrow y = (-1)^3 = -1$ or $y = (1.215)^3 = 1.793$

$$A = \int_{-1}^{1.793} \sqrt[3]{y} - (y^2 - 2)\, dy = \frac{y^{4/3}}{4/3} - \frac{y^3}{3} + 2y \Big|_{-1}^{1.793} = (3.298) - (-0.917) = 4.21$$

We can also use geometry to help us find areas.

Ex. 6) Redo example 3 by removing the triangle formed by the line $y = x - 2$, the x-axis and $x = 4$.

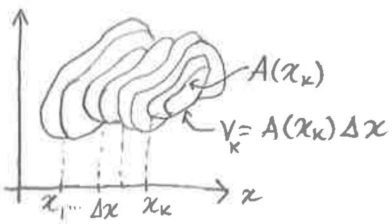
$$A = \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(4-2)(2) = \frac{x^{3/2}}{3/2} \Big|_0^4 - 2 = \frac{2}{3}(8) - 2 = \frac{10}{3}$$

As has been illustrated by the last few examples, you have options when trying to find the area bound by curves. Try to pick the easiest one.

Homework: pg. 380 # 1-35 (odd), 38-43

7.3 Volumes

We can approximate the volume of a solid by breaking it down into "cylinders" with base area A and height Δx and adding together the volumes of these "cylinders".



$V = \sum A(x_k)\Delta x$

This approximation will become accurate once we take the limit as $\Delta x \to 0$. This gives us:

The Volume of a Solid:

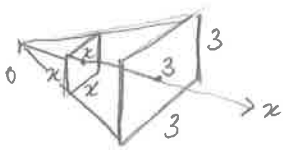
The volume of a solid of known integrable cross-sectional area $A(x)$ on $[a, b]$ is

$$V = \int_a^b A(x)\,dx$$

To find volumes:

1) Sketch the solid and its cross-section
2) Find a formula for $A(x)$
3) Find the limits of integration
4) Integrate $A(x)$ to find the volume

Ex. 1) Find the volume of a square based pyramid of height 3m and base side length 3m.

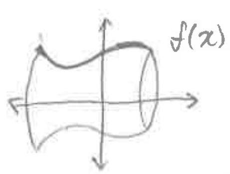


Each cross-section has an area of x^2 for a given x. Since the height is 3m $x \in [0, 3]$.

$$V = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = 9 \text{ m}^3$$

If a graph is rotated in a circle to create a solid, its cross-section is simply a circle whose radius depends on the original graph. Such solids are known as solids of revolution.

Ex. 2) Determine the volume of $f(x) = -x^3 + x + 2$ revolved about the x-axis over $[-1, 1]$.



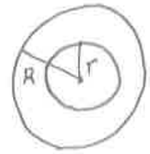
$$A(x) = \pi y^2 = \pi(-x^3+x+2)^2 = \pi(x^6 - 2x^4 - 4x^3 + x^2 + 4x + 4)$$

$$V = \int_{-1}^{1} \pi(x^6 - 2x^4 - 4x^3 + x^2 + 4x + 4)\,dx = \pi\left[\frac{x^7}{7} - \frac{2}{5}x^5 - x^4 + \frac{x^3}{3} + 2x^2 + 4x\right]_{-1}^{1}$$

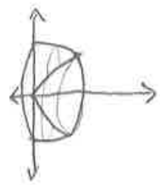
$$= \pi\left[\left(\frac{1}{7} - \frac{2}{5} \cancel{-1} + \frac{1}{3} + 2 + 4\right) - \left(-\frac{1}{7} + \frac{2}{5} \cancel{-1} - \frac{1}{3} \cancel{+2} - 4\right)\right]$$

$$= \pi\left[\frac{2}{7} - \frac{4}{5} + \frac{2}{3} + 8\right] = \frac{856}{105}\pi \doteq 25.6$$

We can also revolve areas bound by curves. Our cross-sectional area is now a washer instead of a circle: $\pi(R^2 - r^2)$.



Ex. 3) The region bound by $x=0$, $y=\cos x$ and $y=\sin x$ is revolved about the x-axis. Find the volume of this solid.



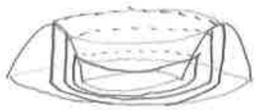
$A(x) = \pi(\cos^2 x - \sin^2 x)$

$0 \le x \le ?$ $\qquad \cos x = \sin x \Rightarrow x = \pi/4 \Rightarrow x \in [0, \pi/4]$

$V = \int_0^{\pi/4} \pi(\cos^2 x - \sin^2 x)\,dx = \pi \int_0^{\pi/4} \cos 2x\,dx = \frac{\pi}{2}\sin 2x \Big|_0^{\pi/4}$

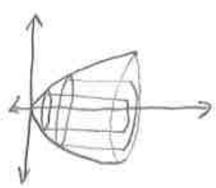
$= \frac{\pi}{2}[1-0] = \frac{\pi}{2}$

Rather than summing cross-sectional areas we can also sum cylindrical shells that grow outward from the axis of revolution.



vs.

Ex. 4) The region bound by $y=\sqrt{x}$, $y=0$, and $x=4$ is revolved about the x-axis. Find the volume of this solid.



The area of each cylindrical shell is $2\pi \underset{\text{radius}}{y}\,\underset{\text{height}}{(4-x)}$

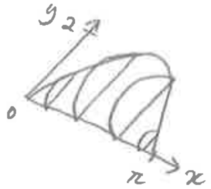
We need to add these areas over the range of $0 \le y \le \sqrt{4} = 2$

$V = \int_0^2 2\pi y(4-y^2)\,dy = 2\pi \int_0^2 (4y - y^3)\,dy = 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2$

$= 2\pi(8-4) = 8\pi$

This method of cross-sectional slicing can be used to find volumes of a wide variety of unusually shaped objects.

Ex. 5) Determine the volume of the solid with a base of the area bound by $y = 2\sin x$ and $y=0$ and with semi-circular cross-sections in the y-direction.



$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{8}d^2 = \frac{\pi}{8}y^2 = \frac{\pi}{8}(2\sin x)^2$

$= \frac{\pi}{2}\sin^2 x \qquad x \in [0, \pi]$

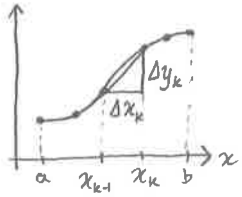
$V = \int_0^{\pi} \frac{\pi}{2}\sin^2 x\,dx = \frac{\pi}{2}\int_0^{\pi}\frac{1}{2}(1-\cos 2x)\,dx = \frac{\pi}{4}\int_0^{\pi}(1-\cos 2x)dx$

$= \frac{\pi}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{4}[(\pi - 0) - (0)] = \frac{\pi^2}{4}$

Homework: pg. 390 # 1, 3-5, 7, 9-11, 13-19, 28, 33, 41, 47, 55 *

7.4 Lengths of Curves

We can estimate the length of a curve by breaking up the curve into straight line segments.



The length of one segment is $\sqrt{\Delta x_k^2 + \Delta y_k^2}$. This can be re-written as $\sqrt{\left(1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right)\Delta x_k^2} = \sqrt{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$. Adding up the segments gives us the total length: $L \approx \sum \sqrt{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$. Taking the limit as $\Delta x_k \to 0$ will give us the actual value for the length:

$$L = \lim_{\Delta x_k \to 0} \sum \sqrt{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$$

$$= \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \because \lim_{\Delta x_k \to 0} \frac{\Delta y_k}{\Delta x_k} = \frac{dy}{dx}$$

Arc Length: Length of a Smooth Curve

Smooth curve begins at (a,c) and ends at (b,d), $a<b$, $c<d$ $\Rightarrow$ (arc) length of the curve is

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

or $$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{[using a similar derivation]}$$

Ex. 1) Find the arc length of $\frac{4\sqrt{2}}{3}x^{3/2} - 1$ from $x=0$ to 1.

$\frac{dy}{dx} = \frac{4\sqrt{2}}{3}\,\frac{3}{2}\,x^{1/2} = 2\sqrt{2}\,x^{1/2}$ $\qquad L = \int_0^1 \sqrt{1+(2\sqrt{2}x^{1/2})^2}\,dx = \int_0^1 \sqrt{1+8x}\,dx$ $\qquad u = 1+8x$, $du = 8dx$

$= \frac{1}{8}\int \sqrt{u}\,du = \frac{1}{8}\,\frac{2}{3}(1+8x)^{3/2}\Big|_0^1 = \frac{1}{12}\left(9^{3/2} - 1^{3/2}\right)$

$= \frac{1}{12}(27-1) = \frac{26}{12} = \frac{13}{6}$

Sometimes using the second version of the arc length formula (integration with respect to y) can be beneficial. Dealing with vertical tangents is one case.

Ex. 2) Find the length of $y = x^{1/3}$ between $(-8,-2)$ and $(8,2)$.

$\frac{dy}{dx} = \frac{1}{3x^{2/3}}$ is not defined at $x=0$ so we cannot integrate there. Instead use

$x = y^3$ $\quad \frac{dx}{dy} = 3y^2 \Rightarrow L = \int_{-2}^{2} \sqrt{1+(3y^2)^2}\,dy \approx 17.26$ (using a calculator)

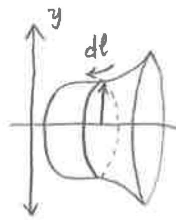
If we are finding the length of a curve with corners or cusps we must split the curve into smooth pieces and add the lengths of those pieces.

Ex. 3) Find the length of the curve $y = x^2 - 4|x| - x$ from $x=-4$ to 4.

y has a corner at $x=0$. Since $y = \begin{cases} x^2+4x-x & \text{if } x<0 \\ x^2-4x-x & \text{if } x \geq 0 \end{cases}$ $\quad y' = \begin{cases} 2x+3 & x<0 \\ 2x-5 & x>0 \end{cases}$ $\quad$ DNE $x=0$

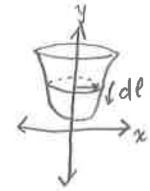
and $L = \int_{-4}^{0} \sqrt{1+(2x+3)^2}\,dx + \int_0^4 \sqrt{1+(2x-5)^2}\,dx \approx 19.56$ (using a calculator)

The surface area of a solid of revolution is found using the arc length formula:



$$A=\int 2\pi f(x)\,dl = \int 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

or $$A=\int 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$$



Ex. 4) Determine the surface area generated by revolving $x=\sqrt{y}$ about the y-axis from $y=0$ to $y=2$.

$g'(y)=\frac{1}{2\sqrt{y}}$ $\quad A=\int_0^2 2\pi\sqrt{y}\sqrt{1+(2\sqrt{y})^{-2}}\,dy = \pi\int_0^2\sqrt{4y\left(1+\frac{1}{4y}\right)}\,dy = \pi\int_0^2\sqrt{4y+1}\,dy$

$u=4y+1 \quad du=4dy \quad y=0 \Rightarrow u=1 \quad y=2\Rightarrow u=9$

$$A=\frac{\pi}{4}\int_1^9 u^{1/2}\,du = \frac{\pi}{4}\,\frac{2}{3}\left[u^{3/2}\right]_1^9 = \frac{\pi}{6}(27-1) = \frac{26\pi}{6} = \frac{13\pi}{3}$$

Homework: pg. 399 # 12, 13, 16, 19, 21, 23, 25, 26, 28, 31, 32

pg. 394 # 58, 61