6.1 Differential Equations and Indefinite Integrals

An equation involving derivatives such as $\frac{dy}{dx} = y \ln x$ is called a <u>differential equation</u>. If y at some x (the <u>initial condition</u>) is given, this becomes an <u>initial value problem</u>. Its solution is called a <u>particular solution</u>. Without an initial condition it is a <u>general solution</u>. The <u>order</u> of the differential equation is indicated by the highest order of derivative in the equation.

Ex. 1 \$100 is invested at 5.6% compounded continuously. Find a formula for the amount in the account at any time.

y is changing by 0.056 of itself at any given time and y starts at 100.

$\therefore \quad \frac{dy}{dt} = 0.056y \quad \& \ y(0) = 100$

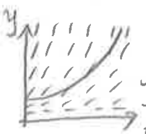
We need a function that is a constant multiple of itself once differentiated. A good guess is $y = Ae^{bt}$ since $y' = Abe^{bt} = by \Rightarrow b = 0.056$.

Thus we have $y = Ae^{0.056t}$. $y(0) = Ae^{0.056(0)} = 100 \Rightarrow A = 100$

$\therefore y = 100e^{0.056t}$ is our particular solution.

Whenever $\frac{dy}{dx} = f(x,y)$ is our differential equation we can plot a <u>slope</u> or <u>direction field</u> by plotting small lines of slope $f(x,y)$ at the points (x,y) since $f(x,y) = dy/dx$ represents the slope of y. Our solution should then flow along this field when plotted on it.

E.g. for ex. 1



$y = 100e^{0.056t}$

$\rightarrow$ these lines have slope $0.056y$

Recall how $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Definition: Indefinite Integral

The set of all antiderivatives of a function $f(x)$ is the indefinite integral of f with respect to x and is denoted by:

$$\int f(x)\,dx$$

Furthermore

$$\int f(x)\,dx = F(x) + C$$

Where C is the constant of integration

E.g. $\int 2x\,dx = x^2 + C$

A Few Indefinite Integrals:

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$ $\qquad \int \frac{dx}{x} = \ln|x| + C$ $\qquad \int e^{kx}\,dx = \frac{e^{kx}}{k} + C$

$\int \sin kx\,dx = -\frac{\cos kx}{k} + C$ $\qquad \int \cos kx\,dx = \frac{\sin kx}{k} + C$ $\qquad \int \sec^2 x\,dx = \tan x + C$

$\int \csc^2 x\,dx = -\cot x + C$ $\qquad \int \sec x \tan x\,dx = \sec x + C$ $\qquad \int \csc x \tan x\,dx = -\csc x + C$

E.g. $\int x^5\,dx = \frac{x^6}{6} + C$ $\qquad \int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{x^{1/2}}{(1/2)} + C = 2\sqrt{x} + C$ $\qquad \int e^{-3x}\,dx = \frac{e^{-3x}}{-3} + C$

$$\int \cos\frac{x}{2}\,dx = \frac{\sin\frac{x}{2}}{(1/2)} + C = 2\sin\left(\frac{x}{2}\right) + C$$

Properties of Indefinite Integrals

$\int k f(x)\,dx = k\int f(x)\,dx \quad k \in \mathbb{R}$
constant multiple rule

$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
sum and difference rule

Ex. 2 Evaluate $\int (x^2 - 2x + 5)\,dx$

$$= \frac{x^3}{3} + C_1 - x^2 + C_2 + 5x + C_3$$

or equivalently $= \frac{x^3}{3} - x^2 + 5x + C$ ← one constant is enough: $C = C_1 + C_2 + C_3$

Ex. 3 A cylindrical tank of radius 2 m and height 4 m drains at the rate of $0.02\sqrt{x}$ m^3/min, where x is the remaining height of fluid remaining. Find a formula for x and find how long it will take for the tank to empty.

The volume at any time is $V = \pi r^2 h = \pi 2^2 x = 4\pi x$.

It drains by $\frac{dV}{dt} = -4\pi \frac{dx}{dt} = 0.02\sqrt{x} \Rightarrow \frac{1}{\sqrt{x}}\frac{dx}{dt} = \frac{-0.02}{4\pi} \Rightarrow \int x^{-1/2}\frac{dx}{dt}\,dt = -\int \frac{0.02}{4\pi}\,dt$

negative because decreasing

$\Rightarrow 2x^{1/2} = -\frac{t}{200\pi} + C \Rightarrow x = \left(\frac{-t}{400\pi} + \frac{C}{2}\right)^2$

When $t = 0$ $x = 4$ $\Rightarrow$ $2\sqrt{4} = 0 + C \Rightarrow C = 4$

$\therefore x = \left(\frac{-t}{400\pi} + 2\right)^2$. The tank empties when $x = 0 \Rightarrow \frac{-t}{400\pi} + 2 = 0 \Rightarrow t = 800\pi$ min.

Ex. 4 Determine formulas for the velocity and position (height) of a rocket fired initially 20 m/s from a height of 4 m.

$a = -9.8\ m/s^2 \Rightarrow \frac{d^2s}{dt^2} = -9.8 \Rightarrow \int \frac{d^2s}{dt^2}\,dt = \int -9.8\,dt \Rightarrow \frac{ds}{dt} = -9.8t + C \Rightarrow 20 = -9.8(0) + C$
$\Rightarrow C = 20$

$\Rightarrow \frac{ds}{dt} = -9.8t + 20 \Rightarrow \int \frac{ds}{dt}\,dt = \int(-9.8t + 20)\,dt \Rightarrow S = -\frac{9.8t^2}{2} + 20t + C$

$\Rightarrow 4 = -\frac{9.8(0)^2}{2} + 20(0) + C \Rightarrow C = 4 \Rightarrow S = -4.9t^2 + 20t + 4$

Homework: pg. 312 # 7, 9, 11, 12, 14, 17–19, 21, 25–37, 40–46, 51, 52, 56–58

6.2 Integration by Substitution

Power Rule for Integration:

u is any differentiable function of x $\Rightarrow$

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

since $\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}$.

Ex. 1 Evaluate $\int (x+2)^2 dx$.

Let $u = x+2 \Rightarrow du = dx \Rightarrow \int (x+2)^2 dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(x+2)^3}{3} + C$

Ex. 2 Evaluate $\int \sqrt{4x-1}\, dx$.

Let $u = 4x-1 \Rightarrow du = 4dx \Rightarrow dx = \frac{du}{4} \Rightarrow \int \sqrt{4x-1}\, dx = \int u^{1/2} \frac{du}{4} = \frac{u^{3/2}}{4(3/2)} + C = \frac{u^{3/2}}{6} + C$

$= \frac{1}{6}(4x-1)^{3/2} + C$

Trigonometric Integral Formulas

$\int \cos u\, du = \sin u + C \qquad \int \sin u\, du = -\cos u + C \qquad \int \sec^2 u\, du = \tan u + C \qquad \int \csc^2 u\, du = -\cot u + C$

$\int \sec u \tan u\, du = \sec u + C \qquad \int \csc u \cot u\, du = -\csc u + C$

Ex. 3 Evaluate $\int \cos(7x+5) dx$

Let $u = 7x+5 \Rightarrow du = 7dx \Rightarrow \int \cos(7x+5) dx = \int \cos u \frac{du}{7} = \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7x+5) + C$

Ex. 4 Evaluate $\int \frac{1}{\cos^2 2x} dx$

Let $u = 2x \Rightarrow du = 2dx \Rightarrow \int \frac{1}{\cos^2 2x} dx = \int \frac{1}{\cos^2 u} \frac{du}{2} = \int \frac{\sec^2 u}{2} du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan 2x + C$

Ex. 5 Evaluate $\int \tan x\, dx$

$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$ Let $u = \cos x \Rightarrow du = -\sin x\, dx \Rightarrow \int \frac{\sin x}{\cos x} dx = \int \frac{-1}{u} du$

$= -\ln|u| + C = \ln\left|\frac{1}{u}\right| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$

In general if we have $\int f(g(x))\, g'(x)\, dx$ and let $u = g(x)$, then $du = g'(x)\, dx$ and

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C.$$

This is known as *integration by substitution*.

For definite integrals this becomes:

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Ex. 6 Evaluate $\int (x^2+2x-3)^2 (x+1)\, dx$

Let $u = x^2+2x-3$, then $du = (2x+2)\, dx = 2(x+1)\, dx \Rightarrow \int (x^2+2x-3)^2 (x+1)\, dx = \int \frac{u^2}{2} du =$

$\frac{u^3}{6} + C = \frac{1}{6}(x^2+2x-3)^3 + C$

Ex. 7 Evaluate $\int \sin^4 x \cos x\, dx$

Let $u = \sin x \Rightarrow du = \cos x\, dx \Rightarrow \int \sin^4 x \cos x\, dx = \int u^4 du = \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C$

Ex. 8 Evaluate $\int_0^{\pi/4} \tan x \sec^2 x \, dx$

Let $u = \tan x \Rightarrow du = \sec^2 x \, dx \Rightarrow \int_0^{\pi/4} \tan x \sec^2 x \, dx = \int_0^1 u \, du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

$u(0) = \tan(0) = 0 \quad u(\frac{\pi}{4}) = 1$

A separable differential equation is one of the form: $\frac{dy}{dx} = g(x)\,h(y)$.

We can separate the variables to solve it:

$$\frac{1}{h(y)} \frac{dy}{dx} = g(x) \Rightarrow \int \frac{1}{h(y)} \frac{dy}{\cancel{dx}} \cancel{dx} = \int g(x)\,dx$$

Ex. 9 Solve $\frac{dy}{dx} = 2x(1+y^2)e^{x^2}$

$$\frac{1}{1+y^2}\frac{dy}{dx} = 2xe^{x^2} \Rightarrow \int \frac{1}{1+y^2}\,dy = \int 2xe^{x^2}dx \Rightarrow \tan^{-1}y + C = \int e^u du = e^u = e^{x^2} \ \left(\substack{\text{one C}\\ \text{is enough}}\right)$$

$$\Rightarrow \tan^{-1}y = e^{x^2} + C \ \left(\substack{\text{Sign of}\\ \text{C is irrelevant}}\right) \Rightarrow y = \tan(e^{x^2} + C)$$

Home work: pg. 321 # 1-6, 8-17, 19-29, 32, 34, 36-44, 49, 51

6.3 Integration by Parts

The product rule states $\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u$. Rearranging and integrating this

becomes $\int u\frac{dv}{dx}dx = \int \frac{d(uv)}{dx}dx - \int v\frac{du}{dx}dx$, which when re-written becomes:

Integration by Parts:

$$\int u\,dv = uv - \int v\,du$$

Ex.1 Evaluate $\int x\cos x\,dx$

Let $u=x$ and $dv=\cos x\,dx \Rightarrow du=dx$ and $v=\sin x \Rightarrow$

$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

We need to choose u and dv strategically to make the integral easier to solve.
E.g. in example 1 if we chose $u=\cos x$ and $dv = x\,dx$ then $du=-\sin x\,dx$ and $v=\frac{x^2}{2}$
and so the integral becomes $\frac{1}{2}x^2\cos x + \int \frac{1}{2}x^2\sin x\,dx$ which is worse than the original.

Ex.2 Determine $\int_0^3 xe^{-x}dx$

Let $u=x$ $dv=e^{-x}dx \Rightarrow du=dx$ $v=-e^{-x} \Rightarrow \int xe^{-x}dx = -xe^{-x}+\int e^{-x}dx$

$= -xe^{-x}-e^{-x}+C \Rightarrow \int_0^3 xe^{-x}dx = [-xe^{-x}-e^{-x}]_0^3 = (-3e^{-3}-e^{-3})-(-e^0) = 1-4e^{-3} \doteq 0.8$

Ex.3 Evaluate $\int \ln x\,dx$

Let $u=\ln x$ $dv=dx \Rightarrow du=\frac{dx}{x}$ $v=x \Rightarrow \int \ln x\,dx = x\ln x - \int \frac{x}{x}dx = x\ln x - x + C$

Ex.4 Evaluate $\int x^2e^x dx$

Let $u=x^2$ $dv=e^xdx \Rightarrow du=2x\,dx$ $v=e^x \Rightarrow \int x^2e^xdx = x^2e^x - \int e^x 2x\,dx$

$\int e^x x\,dx = ?$ Let $u=x$ $dv=e^xdx \Rightarrow du=dx$ $v=e^x \Rightarrow \int e^x x\,dx = xe^x - \int e^xdx = xe^x-e^x$

$\Rightarrow \int x^2e^xdx = x^2e^x - 2(xe^x-e^x)+C = x^2e^x-2xe^x+2e^x+C$

Ex.5 Evaluate $\int e^x\cos x\,dx$

Let $u=e^x$ $dv=\cos x\,dx \Rightarrow du=e^xdx$ $v=\sin x \Rightarrow \int e^x\cos x\,dx = e^x\sin x - \int \sin x\,e^xdx$

$\int e^x\sin x\,dx$? Let $u=e^x$ $dv=\sin x\,dx \Rightarrow du=e^xdx$ $v=-\cos x \Rightarrow \int e^x\sin x\,dx = -e^x\cos x + \int \cos x\,e^xdx$

$\Rightarrow \int e^x\cos x\,dx = e^x\sin x - [-e^x\cos x + \int e^x\cos x\,dx] \Rightarrow 2\int e^x\cos x\,dx = e^x(\sin x+\cos x)$

$\Rightarrow \int e^x\cos x\,dx = \frac{e^x}{2}(\sin x+\cos x)+C$

When using integration by parts repeditively it is usually not a good idea to change your choice of u or dv. Doing so may cause you to undo your work.

Rather than repeat integration by parts many times it is easier to take advantage of a pattern that forms when you do so. This method is known as tabular integration. Consider $\int x^2 e^x\,dx$. Let $f(x) = x^2$ and $g(x) = e^x$ where these are chosen from $u = x^2$ and $dv = e^x\,dx$. Observe the pattern:

f and its derivatives		g and its integrals
		e^x
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0	$-$	

$$\Rightarrow \int x^2 e^x\,dx = x^2 e^x - 2x e^x + 2e^x + C$$

Ex. 6 Evaluate $\int x^3 \sin x\,dx$

Let $f(x) = x^3$ ($\because$ its derivatives will eventually go to zero) and $g(x) = \sin x$.

f(x) and its derivatives		g(x) and its integrals
		$\sin x$
x^3	$+$	$-\cos x$
$3x^2$	$-$	$-\sin x$
$6x$	$+$	$\cos x$
6	$-$	$\sin x$
0	$+$	

$$\Rightarrow \int x^3 \sin x\,dx = -x^3\cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C$$

Homework: pg. 328 # 1-4, 9-13, 15, 18-20, 24, 27-30, 31, 34, 37b*, 40b*

6.4-6.6 Differential Equations Continued

Law of Exponential Change:

$\frac{dy}{dt} = ky \quad y(0) = y_0 \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dt}dt = k\,dt \Rightarrow \int \frac{1}{y}dy = \int k\,dt \Rightarrow \ln|y| = kt + C$

$\Rightarrow e^{\ln|y|} = e^{kt+C} \Rightarrow |y| = e^{kt}e^{C} \Rightarrow y = \pm e^{C}e^{kt} = Ae^{kt} \Rightarrow y_0 = Ae^{k(0)} \Rightarrow A = y_0 \Rightarrow$

$y = y_0 e^{kt}$

Where k is the rate constant: $k>0$ represents growth and $k<0$ represents decay.

Ex. 1 Find the general solution to Newton's Law of Cooling.

$\frac{dT}{dt} = -k(T - T_s) \quad T(0) = T_0 \quad T_s :=$ surrounding temperature $\Rightarrow \frac{d(T-T_s)}{dt} = -k(T - T_s)$

$\therefore d(T-T_s) = dT \Rightarrow (T - T_s) = (T_0 - T_s)e^{-kt}$.

Ex. 2 The resistance a moving object experiences is proportional to its velocity. Find the general solution to the following differential equation:

$m\frac{dv}{dt} = -kv \Rightarrow \frac{dv}{dt} = \left(\frac{-k}{m}\right)v \Rightarrow v = v_0 e^{-\frac{k}{m}t}$.

Ex. 3 How far before a 50 kg skater travelling 7 m/s initially comes to a stop if $k =$ 2.5 kg/s?

$v = 7e^{-(2.5/50)t} \Rightarrow v = 7e^{-0.05t} \Rightarrow S = \int_0^{t_f} v\,dt = \int_0^{t_f} 7e^{-0.05t} = \left[\frac{7e^{-0.05t}}{-0.05}\right]_0^{t_f}$

$= -140e^{-0.05t_f} + 140e^{0} = 140(1 - e^{-0.05t_f}) \qquad t_f = ?$ take $\lim_{t_f \to \infty}$

$S = \lim_{t_f\to\infty} 140(1 - \underbrace{e^{-0.05t_f}}_{\to 0}) = 140$ m.

Logistic Growth:

$\frac{dP}{dt} = \frac{k}{M}P(M-P) \quad P :=$ size of population $M :=$ carrying capacity

Ex. 4 The carrying capacity for a group of bears is 100. The population is now at 10. Determine a logistic model for the population P if $k = 0.1$ and find when the population will be 50.

Substituting: $\frac{dP}{dt} = \frac{0.1}{100}P(100-P) \Rightarrow \frac{1}{P(100-P)}\frac{dP}{dt} = 0.001$ We don't know how to integrate $\frac{1}{P(100-P)}$ so we change it into...

$\frac{A}{P} + \frac{B}{(100-P)} = \frac{1}{P(100-P)} \Rightarrow \frac{A(100-P)+BP}{P(100-P)} = \frac{1}{P(100-P)} \Rightarrow$

$A\,100 - AP + BP = 1 \Rightarrow -AP + BP = 0 \Rightarrow A = B \Rightarrow A = \frac{1}{100} \Rightarrow \frac{1}{100}\left(\frac{1}{P} + \frac{1}{100-P}\right)\frac{dP}{dt} = 0.001$

$B = \frac{1}{100}$

$$\Rightarrow \frac{1}{100}\left(\int \frac{1}{P}\,dP + \int \frac{1}{100-P}\,dP\right) = \int 0.001\,dt \Rightarrow \frac{1}{100}\left(\ln|P| + \int \frac{-1}{u}\,du\right) = 0.001\,t \Rightarrow$$

$u = 100-P$

$du = -dP$

$$\frac{1}{100}\left(\ln|P| - \ln|u|\right) = 0.001\,t + C \Rightarrow \ln|P| - \ln|100-P| = 0.1\,t + C \Rightarrow \ln\left|\frac{P}{100-P}\right| = 0.1t + C$$

$$\Rightarrow \frac{P}{100-P} = \pm e^{0.1t+C} = \pm e^{c} e^{0.1t} = Be^{0.1t} \Rightarrow P = (100-P)Be^{0.1t} \Rightarrow P + PBe^{0.1t} = 100Be^{0.1t}$$

$$\Rightarrow P(1+Be^{0.1t}) = 100Be^{0.1t} \Rightarrow P = \frac{100Be^{0.1t}}{(1+Be^{0.1t})}\,\frac{e^{-0.1t}\frac{1}{B}}{e^{-0.1t}\frac{1}{B}} = \frac{100}{(\frac{1}{B}e^{-0.1t}+1)} \quad \text{Let } A = \frac{1}{B} \Rightarrow$$

$$P = \frac{100}{(1+Ae^{-0.1t})} \qquad P(0) = 10 \Rightarrow 10 = \frac{100}{(1+Ae^{0})} \Rightarrow 10 + 10A = 100 \Rightarrow A = 9 \Rightarrow$$

$$P = \frac{100}{(1+9e^{-0.1t})} \qquad 50 = \frac{100}{(1+9e^{-0.1t})} \Rightarrow 50 + 450e^{-0.1t} = 100 \Rightarrow 450e^{-0.1t} = 50 \Rightarrow$$

$$e^{-0.1t} = \frac{1}{9} \Rightarrow -0.1t = \ln(1/9) \Rightarrow t = \frac{-\ln(1/9)}{0.1} = \frac{\ln(9)}{0.1} \doteq 22 \text{ years.}$$

In general, the solution to $\frac{dP}{dt} = \frac{k}{M}P(M-P)$ is $P = \frac{M}{1+Ae^{-kt}}$, where A is found with the initial condition.

Numerical Solutions:

Say you can't find the solution to a differential equation of the form $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.
Say you can't find the solution to a differential equation using the initial condition and
Instead of solving for y you could estimate the values of y
the value for the slope at a given x value in an interval. This is called a numerical
method and its solution is the numerical solution.

$$\frac{dy}{dx} = f(x_0,y_0) \approx \frac{y(x)-y(x_0)}{(x-x_0)} \Rightarrow y(x) = y(x_0) + f(x_0,y_0)(x-x_0)$$

If $(x-x_0)$ is small, $(x-x_0) \approx dx$. We can then reuse this equation to find our y values numerically. This particular numerical method is called Euler's Method:

$x_1 = x_0 + dx \qquad y_1 = y_0 + f(x_0,y_0)\,dx$

$\vdots \qquad \vdots$

$x_n = x_{n-1} + dx \qquad y_n = y_{n-1} + f(x_{n-1},y_{n-1})\,dx$

The larger the n, the more it strays from the actual y value.

Ex. 5 Find y_1, y_2, y_3 using Euler's method for $y' = 1+y$ $\quad y(0) = 1$ and $dx = 0.1$.

$y_1 = 1 + (1+1)(0.1) = 1.2 \qquad x_1 = 0 + 0.1 = 0.1$

$y_2 = 1.2 + (1+1.2)(0.1) = 1.42 \qquad x_2 = 0.2$

$y_3 = 1.42 + (1+1.42)(0.1) = 1.662 \qquad x_3 = 0.3$

Other numerical methods include the "improved Euler's method": $y_n = y_{n-1} + \underbrace{\left[\frac{f(x_{n-1},y_{n-1}) + f(x_n,z_n)}{2}\right]}_{\text{average between two consecutive slopes}} dx$
where $z_n = y_{n-1} + f(x_{n-1},y_{n-1})\,dx$.

Homework: pg. 339 #23, 25, 29, 31, 38; pg. 347 # 7, 9-12, 17, 21, 25, 27-30, 31*, 32d, 33;
pg. 355 # 1, 5-10