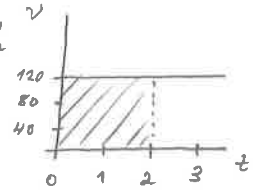
5.1 Rectangular Approximation Method

Say a train travels 120 km/h over a two hour period. How far did the train travel in this time?

$v = \frac{d}{t} \Rightarrow d = vt = (120\ km/h)(2\ h) = 240\ km$ On a v-t graph



this is just the area under the curve for the chosen time period.

We know that the instantaneous velocity is $\frac{dx}{dt}$ so we are going backwards from the derivative by taking the area under the curve. But how do we do this? We will examine three ways to estimate the area under the curve.

Ex. 1 A particle moves with a velocity of $v = t^2$ in a straight line for $t \geq 0$. Where is the particle at $t = 3$?

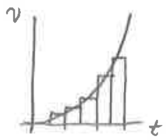
Let us add rectangles of width 0.5s to estimate the area.

Left Rectangle

$v(0) \times 0.5 + v(0.5) \times 0.5 + \ldots + v(2.5) \times 0.5 = (0^2 + 0.5^2 + \ldots + 2.5^2) \times (0.5)$

$= 6.875$ m

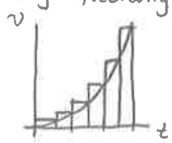
Midpoint Rectangle



$v(0.25) \times 0.5 + v(0.75) \times 0.5 + \ldots + v(2.75) \times 0.5 = (0.25^2 + 0.75^2 + \ldots + 2.75^2) \times (0.5)$

$= 8.9375$ m

Right Rectangle



$v(0.5) \times 0.5 + v(1.0) \times 0.5 + \ldots + v(3.0) \times 0.5 = (0.5^2 + \ldots + 3.0^2) \times (0.5)$

$= 11.375$ m

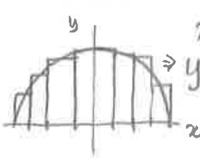
Compare with $d = \frac{1}{3}t^3$

$d = \frac{1}{3}(3)^3$

$= 9$ m

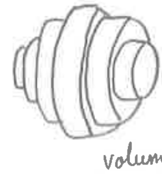
The midpoint rectangular approximation method (MRAM) gives us a middle value so we will use it in the next example.

Ex. 2 Estimate the volume of a solid sphere of radius 4



$x^2 + y^2 = 4^2$

$\Rightarrow y = \sqrt{16 - x^2} \rightarrow$ becomes our r for cylinders with height 1



volume of $\frac{1}{2} \times 2$

$V = \pi r^2 h$

$= \pi(16 - x^2)(1)$

$\Sigma V = 2\pi\left[(16 - 3.5^2) + (16 - 2.5^2) + (16 - 1.5^2) + (16 - 0.5^2)\right]$

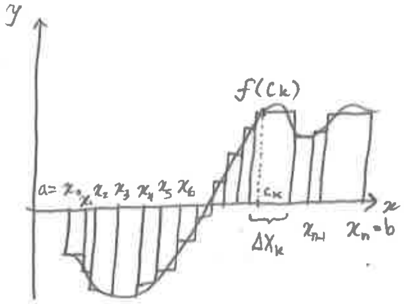
$= 270.2$

Compare with $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4)^3 = 268.1$

Homework: pg. 254 # 1, 2, 4-8, 10, 14 (n = 10 & 20 only)

5.2 Definite Integrals

Let us develop a formalism for the area under a curve:



Let us divide up the x axis into subintervals which produce rectangles of appropriate size for an estimation. The set of boundary points for these rectangles is called the partition of $[a,b]$: $P=\{x_0, x_1, x_2, \dots, x_n\}$. Each subinterval has a length $\Delta x_k = x_k - x_{k-1}$.

The area of the k^{th} rectangle is then $f(c_k)\cdot\Delta x_k$ where c_k is chosen so that $x_{k-1}\le c_k \le x_k$.
The sum of all of these rectangles then becomes:

$$S_n = \sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$$

which is called a Riemann sum for f on the interval $[a,b]$.
This quantity will give us the area under the curve if the lengths of the subintervals tend to zero.
We call this the definite integral.

Definition: The Definite Integral as a Limit of Riemann Sums

Let f be a function defined on $[a,b]$. $\forall$ Partition P of $[a,b]$ let $c_k \in [x_{k-1}, x_k]$

$$\exists\, I : \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I \Rightarrow f \text{ is integrable on } [a,b] \;\&$$

I is the definite integral of f over $[a,b]$

Here $\|P\|$ is the norm of the partition, the longest subinterval length. If it tends to zero, all the other subintervals will as well. The choice of P and the c_k do not matter so long as f is continuous on $[a,b]$.

Theorem: The Existence of Definite Integrals

f continuous on $[a,b] \Rightarrow$ its definite integral exists over $[a,b]$
(i.e. f is integrable over $[a,b]$)

Since the limit tends to the same value regardless of the partition for continuous functions we might as well use regular partitions (subintervals have the same length).

The Definite Integral of a Continuous Function on $[a,b]$

f continuous on $[a,b]$ & $[a,b]$ partitioned into n subintervals of length $\Delta x = (b-a)/n$

$\Rightarrow$ The definite integral of f over $[a,b]$ is

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b f(x)\,dx$$

where $c_k \in [x_{k-1}, x_k]$ is arbitrarily chosen

The Integral of f from a to b:

upper limit, integrand, integral sign, variable of integration, lower limit

$$\int_a^b f(x)\,dx$$

E.g. The definite integral of $f(x) = 3x^2 - 2x + 5$ over the interval $[-1, 3]$ is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(3(c_k)^2 - 2c_k + 5\right)\Delta x = \int_{-1}^{3} 3x^2 - 2x + 5\,dx$$

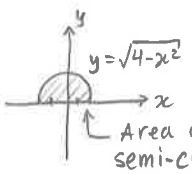
$y = f(x)$ is nonnegative and integrable over $[a,b]$ $\Rightarrow$ area under the curve $y = f(x)$ from a to b is:

$$A = \int_a^b f(x)\,dx$$

Note that if the curve is negative, i.e. $f(x) \le 0$, then $A = -\int_a^b f(x)\,dx$.

Therefore $\int_a^b f(x)\,dx = $ (Area above x-axis) $-$ (Area below x-axis).

Ex. 1 Evaluate $\int_{-2}^{2} \sqrt{4-x^2}\,dx$

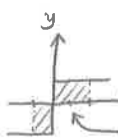


Area of semi-circle of radius 2

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2}\pi(2)^2 = 2\pi$$

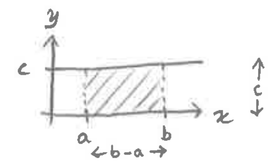
Ex. 2 Evaluate $\int_{-1}^{2} \frac{|x|}{x}\,dx$

$$\frac{|x|}{x} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$



Shaded Area

$$\int_{-1}^{2} \frac{|x|}{x}\,dx = (-1)(1) + (1)(2) = 1$$



Theorem: The Integral of a Constant

$$f(x) = c, \quad c \in \mathbb{R} \text{ on } [a,b] \Rightarrow \int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a)$$

Proof: $\int_a^b c\,dx = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} c\,\frac{b-a}{n} = c(b-a)\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n} = c(b-a)\lim\limits_{n\to\infty} \cancel{n}\left(\frac{1}{\cancel{n}}\right) = c(b-a)$ $\square$

Ex. 3 Determine the distance travelled by a train travelling 110 km/h from 7:00 A.M. to 9:00 A.M.

$$d = \int v\,dt = \int_7^9 110\,dt = 110(9-7) = 220 \text{ km}$$

Homework: pg. 267 # 1–7, 10, 12–23, 29–31, 38, 43, 45, 46

5.3 Definite Integrals and Antiderivatives

Properties:

Order $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Additivity $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Zero $\int_a^a f(x)\,dx = 0$

Max-Min Inequality $(\min f)(b-a) \le \int_a^b f(x)\,dx \le (\max f)(b-a)$

Constant Multiple $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

Domination $f(x) \ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

Sum & Difference $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Ex. 1 If $\int_{-1}^{1} f(x)\,dx = 5$, $\int_1^4 f(x)\,dx = -2$ & $\int_{-1}^{1} h(x)\,dx = 7$, evaluate

(a) $\int_4^1 f(x)\,dx = -\int_1^4 f(x)\,dx = -(-2) = 2$

(b) $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_1^4 f(x)\,dx = 5 + (-2) = 3$

(c) $\int_{-1}^{1} (2f(x) + 3h(x))\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx = 2(5) + 3(7) = 31$

Ex. 2 Show that $\int_0^1 \sqrt{1+\cos x}\,dx < 3/2$

$\max f = \sqrt{1+1} = \sqrt{2} \Rightarrow \int_0^1 f(x)\,dx \le \sqrt{2}\,(1-0) = \sqrt{2} < 3/2$

The Average (Mean) Value:

If f is integrable on $[a,b]$, then its average (mean) value on $[a,b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

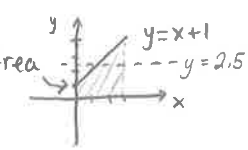
Proof: $av(f) = \dfrac{f_1 + f_2 + \cdots + f_n}{n} = \dfrac{1}{n}\sum_{k=1}^{n} f(c_k) = \dfrac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k)$ $\left[\because \Delta x = \dfrac{b-a}{n} \Rightarrow \dfrac{1}{n} = \dfrac{\Delta x}{b-a}\right]$

$= \dfrac{1}{b-a}\sum_{k=1}^{n} f(c_k)\,\Delta x$ If we take $\lim_{n\to\infty}$, then this becomes $\dfrac{1}{b-a}\int_a^b f(x)\,dx$ $\square$

Ex. 3 Find $av(f)$ for $f(x) = x+1$ on $[0,3]$.

$$av(f) = \frac{1}{3-0}\int_0^3 x+1\,dx = \frac{1}{3}\left(\frac{15}{2}\right) = \frac{5}{2} = 2.5$$

$\int_0^3 x+1\,dx$ ← Area



$= 3(1) + \frac{1}{2}(3)(3)$

$= 3 + \frac{9}{2}$

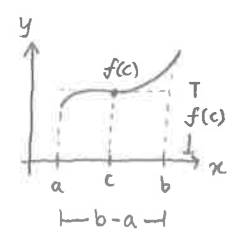
$= \frac{15}{2}$

Notice how $f(1.5) = 2.5 = av(f)$ and $1.5 \in [0,3]$. This happens in general.

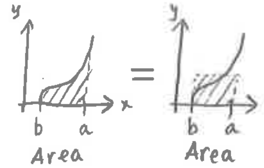
Theorem: The Mean Value Theorem for Definite Integrals

f continuous on $[a,b] \Rightarrow \exists\, c \in [a,b]$:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

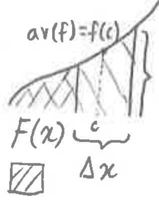


Note that these results mean that $\int_a^b f(x)\,dx = av(f)\,(b-a)$, i.e. that the area under $f(x)$ is equal to the area under $y = av(f)$.



In the next section we will see how to find integrals without having to measure the area under the curve but here is a taste.

Let $F(x)$ represent the area under the curve from a reference point to x. Then we get:



$F(x+\Delta x)$ $\qquad (c \in [x, x+\Delta x])$

$$F(x+\Delta x) - F(x) = av(f)\,\Delta x \quad \Rightarrow \quad av(f) = f(c) = \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

Now take $\lim\limits_{\Delta x \to 0}$:
$$\lim_{\Delta x \to 0} f(c) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$
$$f(x) = \frac{d}{dx} F(x) \quad \Rightarrow \quad \boxed{f(x) = \frac{d}{dx}\int_a^x f(t)\,dt}$$

We have essentially proved what is known as <u>The Fundamental Theorem of Calculus</u> which links derivatives and integrals: they are opposite operations.

Say we had a function $F(x)$ and $\frac{dF(x)}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. The derivative of $F(x)$ gives us $f(x)$ and $F(x) = \int_a^x f(t)\,dt$. So if we can find a function $F(x)$ which when differentiated becomes $f(x)$, we can find the integral. For obvious reasons we call $F(x)$ the <u>antiderivative</u> of f. Since the derivative of a constant C is zero we can add it to $F(x)$:
$$\int_a^x f(t)\,dt = F(x) + C$$
$$x = a \Rightarrow \int_a^a f(t)\,dt = 0 = F(a) + C \quad \Rightarrow \quad C = -F(a) \quad \Rightarrow \quad \boxed{\int_a^x f(t)\,dt = F(x) - F(a)}$$

Ex. 4 Find $\int_0^{\pi} \sin x\,dx$

$$\frac{dF(x)}{dx} = \sin x \quad \Rightarrow \quad F(x) = -\cos x \quad \Rightarrow \quad \int_0^{\pi} \sin x\,dx = \big(-\cos(\pi)\big) - \big(-\cos(0)\big) = -(-1) - (-1)$$
$$= 1 + 1 = 2$$

Homework: pg. 274 # 1, 2, 7-10, 13 – 21, 27, 33, 35, 39, 40, 42, 46

5.4 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part I:

f is continuous on $[a,b] \Rightarrow F(x) = \int_a^x f(t)\,dt$ has a derivative $\forall x \in [a,b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof: $\frac{dF}{dx} = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \lim_{h\to 0} \frac{\int_x^a f(t)dt + \int_a^{x+h} f(t)dt}{h}$

$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)dt}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c)$ [where $c \in [x, x+h]$ by mean value theorem]

$\because c \in [x, x+h]$ as $h \to 0$, $c \to x$

$\Rightarrow \frac{dF}{dx} = \lim_{h\to 0} f(c) = f(x)$ $\square$

E.g. $\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$

Ex.1 Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t\,dt$ let $u = x^2$ $\quad \frac{d}{du}\int_1^u \cos t\,dt = \frac{dy}{du} = \cos u$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \cdot 2x = 2x\cos x^2$

Ex.2 Find $\frac{d}{dx}\int_{2x}^{x^2} \frac{1}{2+e^t}\,dt$

$\frac{d}{dx}\int_{2x}^{x^2} \frac{1}{2+e^t}\,dt = \frac{d}{dx}\left(\int_{2x}^0 f(t)\,dt + \int_0^{x^2} f(t)dt\right) = \frac{d}{dx}\left(\int_0^{x^2} f(t)dt - \int_0^{2x} f(t)dt\right)$

$= \frac{1}{2+e^{x^2}}\cdot 2x - \frac{1}{2+e^{2x}}\cdot 2 = \frac{2x}{2+e^{x^2}} - \frac{2}{2+e^{2x}}$

chain rule

The Fundamental Theorem of Calculus, Part II (The Integral Evaluation Theorem):

f is continuous on $[a,b]$ and $\frac{dF(x)}{dx} = f(x)$ [$F(x)$ is any antiderivative of f] on $[a,b] \Rightarrow$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof: Say $G(x) = \int_a^x f(t)dt$ Let $F(x) = G(x) + C$ for some $C \in \mathbb{R}$, then

$F(b) - F(a) = G(b) + C - (G(a)+C) = G(b) - G(a) = \int_a^b f(t)\,dt + \underbrace{\int_a^a f(t)\,dt}_{0} = \int_a^b f(t)\,dt$ $\square$

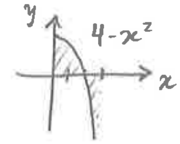
Ex.3 Evaluate $\int_{-1}^3 (x^3+1)\,dx$

We need to find $F(x)$ such that $\frac{dF(x)}{dx} = x^3+1$. $\quad F(x) = \frac{x^4}{4} + x$

$\Rightarrow \int_{-1}^3 x^3+1\,dx = F(3) - F(-1) = \left(\frac{3^4}{4}+3\right) - \left(\frac{(-1)^4}{4} + (-1)\right) = 24$

Ex. 4 Find the area between the curve $f(x) = 4 - x^2$; $0 \le x \le 3$ and the x-axis.

$f(x)$ is above the x-axis when $0 \le x < 2$ and below it when $2 < x \le 3$ since $4 - x^2 = 0 \Rightarrow x = \pm 2$. We need to add these two areas separately as the integral will subtract the areas under the x-axis from those above:



$$\int_0^2 4 - x^2\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

$$\int_2^3 4 - x^2\,dx = \left[4x - \frac{x^3}{3}\right]_2^3 = 12 - \frac{27}{3} - \frac{16}{3} = -\frac{7}{3}$$

Total Area $= \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$

Strategy for Finding Total Area:

1) Find the zeros of f and partition $[a, b]$ with them.
2) Integrate f over each subinterval.
3) Add the absolute values of the integrals.

Ex. 5 The cost for producing the first 10 units is \$200. afterwards the marginal cost at x units output is $\frac{dc}{dx} = \frac{1000}{x^2}$. Find the total cost for producing 100 units.

$$c(100) = 200 + c(100) - c(10) = 200 + \int_{10}^{100} \frac{dc}{dx}\,dx = 200 + \int_{10}^{100} \frac{1000}{x^2}\,dx$$

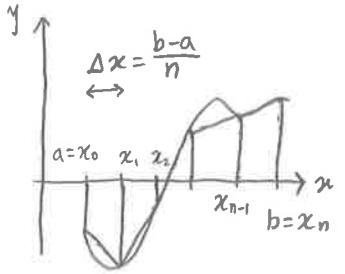
$$= 200 + \left[\frac{-1000}{x}\right]_{10}^{100} = 200 + \left[\frac{-1000}{100} - \left(\frac{-1000}{10}\right)\right] = 200 + 90 = 290$$

$\therefore$ The cost for producing 100 units is \$290.

Homework: pg. 286 # 1-16, 18-20, 25-28, 35, 37-39, 41, 43-48, 50, 51, 55, 57

5.5 The Trapezoidal Rule

Recall how we used rectangles to approximate the area under the curve. More accurate values can be obtained by using trapezoids instead of rectangles.



$$\int_a^b f(x)\,dx \approx \Delta x \frac{f(x_0)+f(x_1)}{2} + \Delta x \frac{f(x_1)+f(x_2)}{2} + \dots + \Delta x \frac{f(x_{n-1})+f(x_n)}{2}$$

$$= \frac{\Delta x}{2}\Big[f(x_0) + \underbrace{f(x_1) + f(x_1)}_{2f(x_1)} + \underbrace{f(x_2) + \cdots}_{2f(x_2)} + \underbrace{\cdots f(x_{n-1})}_{2f(x_{n-1})} + f(x_n) \Big]$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n) \right]$$

$$= \frac{LRAM_n + RRAM_n}{2}$$

Ex. 1 Use the trapezoidal rule to estimate $\int_1^2 x^2 dx$ with $n = 4$.

$$\int_1^2 x^2 dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right] \qquad \Delta x = \frac{2-1}{4} = \frac{1}{4}$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)\left[(1)^2 + 2(1.25)^2 + 2(1.5)^2 + 2(1.75)^2 + (2)^2 \right]$$

$$= \frac{75}{32} = 2.34375$$

Compare this with the actual area of

$$\int_1^2 x^2 dx = \left[\frac{x^3}{3} \right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2.3\overline{3}$$

Ex. 2 Calculate the average temperature for the following data set.

Time	12	13	14	15	16	17	18	19	20	21	22	23	24
Temperature	63	65	66	68	70	69	68	68	65	64	62	58	55

This is a function with given data so $\quad av(T) = \frac{1}{24-12}\int_{12}^{24} T(t)\,dt$

but we cannot solve for the integral so

$$av(T) = \frac{1}{12}\int_{12}^{24} T(t)\,dt \approx \frac{1}{12}\,\frac{(1)}{2}\left[T(12) + 2T(13) + \dots + 2T(23) + T(24) \right]$$

$$= \frac{1}{24}\left[63 + 2(65) + \dots + 2(58) + 55 \right] = \frac{1}{24}\left[1564 \right] \doteq 65.17\,°F$$

We can also approximate for the area under the curve using parabolic sections. This method is formulated in Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$$

where $[a, b]$ is partitioned into an <u>even</u> number n of subintervals of length $\Delta x = \frac{b-a}{n}$ and $a = x_0$ and $b = x_n$.

Ex. 3 Use Simpson's Rule to approximate $\int_0^2 5x^4\,dx$ with $n=4$.

$$\int_0^2 5x^4\,dx \approx \left(\frac{2-0}{4}\right)\left(\frac{1}{3}\right)\left[5(0)^4 + 4(5)(0.5)^4 + 2(5)(1)^4 + 4(5)(1.5)^4 + 5(2)^4\right]$$

$$= \left(\frac{1}{6}\right)(192.5) = 32.08\overline{3} = \frac{385}{12}$$

Compare with $\int_0^2 5x^4\,dx = x^5\Big|_0^2 = 2^5 - 0^5 = 32$

Homework: pg. 295 # 1-8, 10a, 11, 23