4.1 Extreme Values of Functions

Absolute (Global) Extrema:

f is a function with domain $D \Rightarrow f(c)$ (a) absolute maximum on $D \Leftrightarrow f(x) \le f(c)\ \forall x \in D$
(b) absolute minimum on $D \Leftrightarrow f(x) \ge f(c)\ \forall x \in D$

Ex. 1 Determine any global extrema for the following:

(a) $y = \sin x$ on $[0, 2\pi]$ 1 when $x = \pi/2$ and -1 when $x = 3\pi/2$

(b) $y = x^2$ on $(0, 2]$ no min, max of $2^2 = 4$

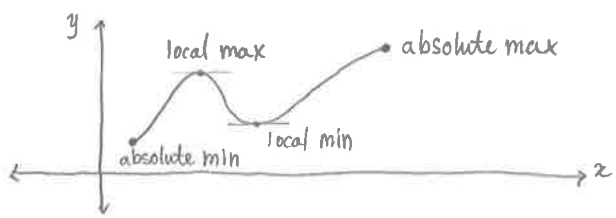
(c) $y = x^3 + 2$ on $[0, 1)$ no max, min of 2

The Extreme Value Theorem:

f continuous on $[a,b] \Rightarrow f$ has both a maximum and minimum on $[a,b]$

Local Extrema:

c is an interior point of the domain of $f \Rightarrow f(c)$ (a) local maximum $\Leftrightarrow f(x) \le f(c)\ \forall x \in$ open interval about c
(b) local minimum $\Leftrightarrow f(x) \ge f(c)\ \forall x \in$ open interval about c



Theorem: Local Extreme Values

f has a local max/min at an interior point c of its domain & $f'(c)$ exists $\Rightarrow f'(c) = 0$

We call points c where $f'(c) = 0$ or does not exist critical points.

Extrema only occur at critical points and endpoints.

Ex. 2 Find the extrema of $f(x) = x^{2/3}$ on $[-2, 3]$.

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$ critical points $\Rightarrow 0 \ne \frac{2}{3\sqrt[3]{x}}$ $3\sqrt[3]{x} = 0$ when $x = 0$

$f(0) = 0$

end points $f(-2) = \sqrt[3]{(-2)^2} = \sqrt[3]{4} \doteq 1.6$

$f(3) = \sqrt[3]{9} \doteq 2.1$

$\therefore$ (abs.) min @ $(0,0)$ and (abs.) max @ $(3, \sqrt[3]{9})$

Ex. 3 Find the extrema of $f(x) = \frac{1}{\sqrt{4-x^2}}$. $f'(x) = -\frac{1}{2}(4-x^2)^{-3/2}(-2x) = \frac{x}{\sqrt{4-x^2}^{\,3}}$

critical points $x = 0$ $4 - x^2 = 0 \Rightarrow 4 = x^2 \Rightarrow x = \pm 2$

$f(-3) = DNE$ $f(-2) = DNE$ (∞) $f(-1) \doteq 0.6$ $f(0) = 0.5$ $f(1) \doteq 0.6$ $f(2) = DNE$ (∞) $f(3) = DNE$

$\uparrow$ min (abs.)

Note that not every critical point or endpoint is an extreme value. E.g. $\frac{d}{dx}x^3 = 3x^2$ which is 0 when $x=0$ but $x=0$ is not an extremum on x^3 (it is a "point of inflection").

Ex. 4 Find the extrema of $f(x) = \begin{cases} 5-2x^2 & x \le 1 \\ x+2 & x > 1 \end{cases}$

$f'(x) = \begin{cases} -4x & x<1 \\ 1 & x>1 \end{cases}$ $\quad -4x = 0 \Rightarrow x=0 \quad f(0) = 5 \quad f(-0.1) = 4.98 = f(0.1)$

max

$f(1) = 3 \quad f(0.9) = 3.38 \quad f(1.1) = 3.1 \quad f(-3) = -13 < 3 \quad f(10) = 12 > 5$

min

$\therefore$ (0,5) is a local max. and (1,3) is a local min.

Ex. 5 Find the extrema of $f(x) = \ln\left|\frac{x}{1+x^2}\right| = \begin{cases} \ln\left(\frac{-x}{1+x^2}\right) & x<0 \\ \ln\left(\frac{x}{1+x^2}\right) & x>0 \end{cases}$

$$f'(x) = \frac{1}{\left(\frac{-x}{1+x^2}\right)}\left(\frac{-(1+x^2)-2x(-x)}{(1+x^2)^2}\right) = \frac{1-x^2}{x(1+x^2)} \text{ in both situations}$$

$$= \frac{1}{\left(\frac{x}{1+x^2}\right)}\left(\frac{(1+x^2)-(2x)(x)}{(1+x^2)^2}\right)$$

critical points $\quad 0 = 1-x^2 \Rightarrow x = \pm 1$

$0 = x(1+x^2) \Rightarrow x=0$ but $0 \notin \text{Domain}(f(x))$

$f(-\tfrac{1}{2}) \doteq -0.9 \quad f(-1) \doteq -0.7 \quad f(-2) \doteq -0.9$
$f(\tfrac{1}{2}) \doteq \quad f(1) \doteq \quad f(2) \doteq$

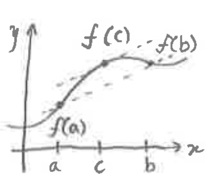
(abs.) max

Homework: pg. 184 # 1, 3, 6, 8, 10-12, 14, 15, 17, 19, 21, 23, 24, 26, 27, 30, 38, 39, 42, 45-49

4.2 Mean Value Theorem

Mean Value Theorem:

$f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) $\Rightarrow$ $\exists c \in (a,b) : f'(c) = \dfrac{f(b)-f(a)}{b-a}$



← slope : $f'(c)$

← slope : $\dfrac{f(b)-f(a)}{b-a}$

Consider a car accelerating from zero which takes 8 s to go 120 m. Its <u>average</u> velocity is 120 m / 8 s = 15 m/s. As its average, it makes sense that the velocity at one point is ≥ 15 m/s. Mean value theorem says at one point the instantaneous velocity <u>is</u> 15 m/s. Since v will be ≥ 15 m/s eventually this conclusion is obvious in this case.

Ex. 1 Find where the mean value theorem is satisfied for $f(x) = \sqrt{1-x^2}$ on $(-1, 1)$.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{(-2x)}{2\sqrt{1-x^2}} = \frac{f(1)-f(-1)}{1-(-1)} \nearrow \frac{-x}{\sqrt{1-x^2}} = \frac{0-0}{2} \qquad \therefore \ c = 0$$

$$x = 0$$

f is defined on I and $a, b \in I$

f increases on I $\Leftrightarrow$ $[a < b \Rightarrow f(a) < f(b)]$

f decreases on I $\Leftrightarrow$ $[a < b \Rightarrow f(a) > f(b)]$

Corollary:

f is continuous on $[a,b]$ and differentiable on (a,b)

$f'(x) > 0 \quad \forall x \in (a,b) \Rightarrow f$ increases on $[a,b]$

$f'(x) < 0 \quad \forall x \in (a,b) \Rightarrow f$ decreases on $[a,b]$

Proof: Let $x_1, x_2 \in [a,b]$ $x_1 < x_2$. Mean value theorem $\Rightarrow \exists c \in (a,b) : f'(c) = \dfrac{f(x_2)-f(x_1)}{x_2 - x_1}$

$\Leftrightarrow \quad (x_2 - x_1) f'(c) = f(x_2) - f(x_1) \qquad x_1 < x_2 \Leftrightarrow (x_2 - x_1) > 0$

$f' > 0 \Rightarrow f(x_2) - f(x_1) > 0 \Leftrightarrow f(x_2) > f(x_1) \Leftrightarrow f$ increases on $[a,b]$

$f' < 0 \Rightarrow f(x_2) - f(x_1) < 0 \Leftrightarrow f(x_2) < f(x_1) \Leftrightarrow f$ decreases on $[a,b]$ $\square$

Ex. 2 Determine the intervals of increase and decrease for $f(x) = x^3 - 4x$

$f'(x) = 3x^2 - 4 \qquad f' = 0 \qquad 3x^2 - 4 = 0 \qquad (\sqrt{3}x - 2)(\sqrt{3}x + 2) = 0 \qquad x = \dfrac{2}{\sqrt{3}}$ or $x = -\dfrac{2}{\sqrt{3}}$

critical points

		$-2/\sqrt{3}$		$2/\sqrt{3}$	
f'	+		−		+
↘/↗	↗		↘		↗

$\therefore$ $f(x)$ is increasing when $x < -2/\sqrt{3}$ or $x > 2/\sqrt{3}$ and decreasing when $-2/\sqrt{3} < x < 2/\sqrt{3}$.

Corollary:

$f'(x) = 0 \quad \forall x \in I \Rightarrow \exists C \in \mathbb{R} : f(x) = C \quad \forall x \in I$

Proof: $a, b \in I$, $b > a$ $\quad \dfrac{f(b) - f(a)}{b - a} = f'(c) = 0 \Rightarrow f(b) = f(a) \Rightarrow f(x) = C \quad C \in \mathbb{R}$ $\square$

Corollary:

$f'(x) = g'(x) \quad \forall x \in I \Rightarrow \exists C \in \mathbb{R} : f(x) = g(x) + C \quad \forall x \in I$

Proof: $h(x) = f(x) - g(x) \quad h'(x) = f'(x) - g'(x) = 0 \Rightarrow h(x) = C \quad (C \in \mathbb{R}) \Rightarrow f(x) - g(x) = C$

$\Rightarrow f(x) = g(x) + C$ $\square$

$F(x)$ is the antiderivative of $f(x) \Leftrightarrow F'(x) = f(x) \quad \forall x \in \text{domain}(f)$

Ex. 3 Find the function $f(x)$ whose derivative is $\sin x$ and passes through the point $(0, 2)$.

$f(x) = -\cos x + C \quad (\because f'(x) = \sin x) \qquad f(0) = -\cos(0) + C = 2$

$-1 + C = 2$

$C = 3$

$\therefore f(x) = -\cos x + 3$

We can find the kinematics equations from physics this way:

$a = 9.8 \text{ m/s}^2 = g$

$v = gt + C \quad (\because v'(t) = a(t) = g) \qquad C = v_0 \quad \text{so} \quad v = gt + v_0$

$x = \frac{1}{2}gt^2 + v_0 t + C \quad (\because x'(t) = v(t) = gt + v_0) \quad C = x_0 \quad \text{so} \quad x = \frac{1}{2}gt^2 + v_0 t + x_0$

Homework: pg. 192 # 2, 4, 5, 7, 10, 12, 14, 15, 17, 18, 25, 27, 29 – 31, 34, 39

4.3 First and Second Derivative Tests

The First Derivative Test for Local Extrema:

at a critical point c:

$f'>0$ ↗ ↘ $f'<0$; $x<c$ c $x>c$ $\Rightarrow f(c)$ is a local max

$f'<0$ ↘ ↗ $f'>0$; $x<c$ c $x>c$ $\Rightarrow f(c)$ is a local min

$f'>0$ ↗ ↗ $f'>0$; $x<c$ c $x>c$

$f'<0$ ↘ ↘ $f'<0$; $x<c$ c $x>c$

$\Rightarrow$ no extremum at $f(c)$

at a left endpoint a:

↗ $f'>0$; a $x>a$ $\Rightarrow f(a)$ is a local min

↘ $f'<0$; a $x>a$ $\Rightarrow f(a)$ is a local max

at a right endpoint b:

↘ $f'<0$; $x<b$ b $\Rightarrow f(b)$ is a local min

↗ $f'>0$; $x<b$ b $\Rightarrow f(b)$ is a local max

Ex. 1 Determine the extrema and intervals of increase and decrease for $f(x)=x^3-12x-5$.

Critical points: $f'(x)=3x^2-12=0 \Rightarrow x=\pm 2$

First Derivative Test: f': $+$ ↗ | -2 (max) | $-$ ↘ | 2 (min) | $+$ ↗

local max @ $x=-2$ increasing $x<-2$ $x>2$

local min @ $x=2$ decreasing $-2<x<2$

Ex. 2 $f(x)=(x^2-3)e^x$.

Critical points: $f'(x)=2xe^x+e^x(x^2-3)=e^x(2x+x^2-3)=0 \Rightarrow x^2+2x-3=0$

$\Rightarrow (x+3)(x-1)=0 \Rightarrow x=-3$ or $x=1$

$|x|>\sqrt{3} \Rightarrow f(x)>0 \Rightarrow$ global min @ $x=1$

First Derivative Test: f': $+$ ↗ | -3 (max) | $-$ ↘ | 1 (min) | $+$ ↗

local max @ $x=-3$ increasing $x<-3$ $x>1$

decreasing $-3<x<1$

Concavity:

$f(x)$ is:

<u>Concave up</u> if y' is increasing or $y''>0$ $\cup$

<u>concave down</u> if y' is decreasing or $y''<0$ $\cap$

E.g. $y=x^2$ is concave up as $y''=2>0$ always

$y=\sin x$ is concave up on $(\pi,2\pi)$ as $y''=-\sin x>0$ on $(\pi,2\pi)$

and concave down on $(0,\pi)$ as $y''<0$ on $(0,\pi)$

Point of Inflection:

A point (with a tangent line) where the concavity changes.

Ex. 3 Describe the motion of a particle moving according to $s(t) = 2t^3 - 14t^2 + 22t - 5$

Critical points: $6t^2 - 28t + 22 = 0$

$2(3t^2 - 14t + 11) = 0$

$2(3t^2 - 11t - 3t + 11) = 0$

$2(t(3t-11) - (3t-11)) = 0$

$2(t-1)(3t-11) = 0$

$t = 1$ or $t = 11/3$

First Derivative Test:

		1		$11/3$	
f'	$+$		$-$		$+$
	$\nearrow$		$\searrow$		$\nearrow$
		max		min	
	forwards		backwards		forwards

$f'' = 12t - 28 = 0 \Rightarrow t = \frac{28}{12} = \frac{7}{3}$

		$7/3$	
f''	$-$		$+$
	$\cap$		$\cup$
	$-$'ve a		$+$'ve a

Overall picture: decelerating forwards to $t = 1$

accelerating backwards to $t = 7/3$

decelerating backwards to $t = 11/3$

accelerating forwards afterwards

Second Derivative Test:

$f'(c) = 0$ & $f''(c) < 0 \Rightarrow$ local max @ $x = c$

$f'(c) = 0$ & $f''(c) > 0 \Rightarrow$ local min @ $x = c$

Note that this test fails if $f''(c) = 0$ or if it does not exist. Use the first derivative test if this happens.

Ex. 4 Find the extrema of $f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12 = 0 \Rightarrow x = \pm 2$ are critical points

$f''(x) = 6x$

$f''(-2) = -12 < 0 \Rightarrow$ (local) max

$f''(2) = 12 > 0 \Rightarrow$ (local) min

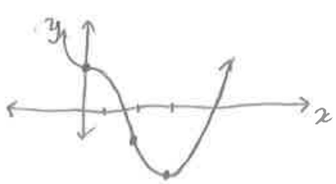
Ex. 5 If $f'(x) = 4x^3 - 12x^2$, analyze $f(x)$ & graph it.

$4x^3 - 12x^2 = 0 \Rightarrow x^2(4x - 12) = 0 \Rightarrow x = 0$ or $x = 3$ are critical points

		0		3	
f'	$-$		$-$		$+$
	$\searrow$		$\searrow$		$\nearrow$
				min	

$f'' = 12x^2 - 24x = x(12x - 24) = 0 \Rightarrow x = 0$ or $x = 2$

		0		2	
f''	$+$		$-$		$+$
	$\cup$		$\cap$		$\cup$
		infl.		infl.	



Homework: pg. 203 # 1, 3, 8-12, 15, 17, 21, 22, 24, 28, 29, 31, 33, 39, 41, 46, 47, 52*

4.4 Optimization

Ex. 1 An open box is made by removing squares of side x from a 25" by 20" piece of cardboard. What value of x creates the box with the maximum volume?

$$V(x) = (20-2x)(25-2x)x = 4x^3 - 90x^2 + 500x \qquad 0 < x < 10$$

$$V'(x) = 12x^2 - 180x + 500 = 0 = 4(3x^2 - 45x + 125)$$

$$x = \frac{45 \pm \sqrt{45^2 - 4(3)(125)}}{2(3)} = \frac{45 \pm \sqrt{525}}{6} \qquad x \doteq 11.3 \text{ or } x \doteq 3.7$$

$$V''(x) = 24x - 180 \qquad \underset{\substack{\text{min} \\ [\text{not in domain}]}}{V''(11.3) = 91.6 > 0} \qquad \underset{\text{max}}{V''(3.7) \doteq -91.6 < 0}$$

$\therefore$ V_{max} occurs at $x = 3.7$" and is $V(3.7) = 820.53 \text{ in}^3$

Ex. 2 Determine the dimensions of a 1 L cylinder made with the least amount of material possible.

$$V(r) = \pi r^2 h = 1000 \text{ cm}^3 \Rightarrow h = \frac{1000}{\pi r^2}$$

$$A(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

$$A'(r) = 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow 4\pi r = \frac{2000}{r^2} \Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$$

$$A''(r) = 4\pi + \frac{4000}{r^3} \qquad A''\left(\sqrt[3]{\frac{500}{\pi}}\right) = 4\pi + 4000\left(\frac{\pi}{500}\right) = 12\pi > 0 \Rightarrow \text{min}$$

$$\therefore \quad r = \sqrt[3]{\frac{500}{\pi}} \doteq 5.4 \text{ cm} \qquad h = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}} = \sqrt[3]{\frac{1000^3\pi^2}{\pi^3 500^2}} = \sqrt[3]{\frac{4 \cdot 1000}{\pi}} = \sqrt[3]{\frac{8 \cdot 500}{\pi}} = 2\sqrt[3]{\frac{500}{\pi}}$$

$$= 2r \doteq 10.8 \text{ cm}$$

$A_{min} = 2\pi r(r+h) = 2\pi(5.4)(5.4+10.8) \doteq 550 \text{ cm}^2$ is achieved with $r = 5.4$ cm & $h = 10.8$ cm.

Ex. 3 Find the maximum possible product of two numbers whose sum is 20.

$$P(x) = x(20-x) = 20x - x^2 \qquad P'(x) = 20 - 2x = 0 \Rightarrow x = 10 \qquad \underset{\text{max}}{P''(10) = -2 < 0}$$

$\therefore$ The two numbers are 10 & 10.

Ex. 4 What is the largest rectangle that can be inscribed under the sine curve.

$$l = \pi - 2x \qquad 0 < x < \pi/2 \qquad A = (\pi - 2x)\sin x \qquad A'(x) = -2\sin x + \cos x(\pi - 2x)$$

$$h = \sin x$$

$0 = -2\sin x + \cos x(\pi - 2x)$ cannot be solved algebraically

Using technology $x \doteq 0.71$ so $l \doteq 1.72$ and $h \doteq 0.65$, giving an area of 1.12.

Applications to Economics:

$r(x)$ = revenue $\quad p(x) = r(x) - c(x)$ = profit $\Big\}$ as a function of x items
$c(x)$ = cost

$\frac{dr}{dx}$ = marginal revenue $\quad \frac{dc}{dx}$ = marginal cost $\quad \frac{dp}{dx}$ = marginal profit

Profit is maximized when $p'(x) = r'(x) - c'(x) = 0$, so when $r'(x) = c'(x)$

Ex. 5 Suppose $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$ where x is in thousands. When is profit maximized?

$$r'(x) = 9 = c'(x) = 3x^2 - 12x + 15 \Rightarrow 3x^2 - 12x + 6 = 0 \Rightarrow 3(x^2 - 4x + 2) = 0$$

$$\Rightarrow x = \frac{4 \pm \sqrt{4^2 - 4(2)}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2} \qquad p''(x) = -6x + 12$$

$$p''(2+\sqrt{2}) \doteq -8.5 < 0 \Rightarrow \text{max} \qquad p''(2-\sqrt{2}) \doteq 8.5 > 0 \Rightarrow \text{min}$$

$\therefore$ a max profit of $p(2+\sqrt{2}) = 9.6$ occurs when $(2+\sqrt{2} =)\ 3.4$ thousand units are produced and sold.

Average cost is defined as $\frac{c(x)}{x}$. It is minimized when $\frac{c'(x)x - c(x)}{x^2} = 0 \Rightarrow$
$c'(x)x - c(x) = 0 \Rightarrow \frac{c(x)}{x} = c'(x)$. I.e. when the average cost equals the marginal cost.

Ex. 6 Minimize the average cost if $c(x) = x^3 - 6x^2 + 15x$, where x is in thousands.

$$c'(x) = 3x^2 - 12x + 15 = \frac{c(x)}{x} = x^2 - 6x + 15 \Rightarrow 2x^2 - 6x = 0 \Rightarrow x(2x - 6) = 0$$

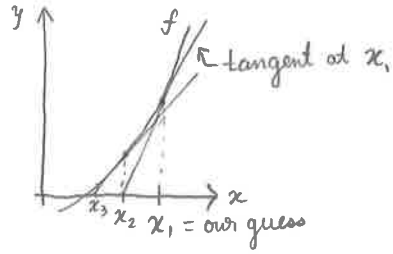
$$\Rightarrow x = 0 \text{ or } x = 3 \qquad \left(\frac{c(x)}{x}\right)' = 2x - 6 \Rightarrow \left(\frac{c(x)}{x}\right)'' = 2 > 0 \Rightarrow \text{both are mins}$$

Cost is obviously minimal when $x = 0$, we are interested only when $x = 3$. So the minimal average cost occurs when 3 thousand units are produced and is $\frac{c(3)}{3} = 6$.

Homework: pg. 214 # 1-7, 9-11, 13-18, 20, 22, 31, 35, 36, 42, 45, 49, 52

4.5 Newton's Method & Differentials

Newton's method is a way to approximate a zero of a function.
Say we guess the zero at first. We can use the tangent line of the guess to get us closer to the zero. This procedure can then be repeated to get as close to the zero as we desire.



If $f'(a)$ is the slope of f at $x=a$, then the tangent can be generalized as:

$$y = f'(a)\,x + b$$

$$f(a) = f'(a)\,a + b$$

$$\Rightarrow b = f(a) - a f'(a)$$

$$\Rightarrow y = f'(a)\,x + f(a) - a f'(a)$$

$$\boxed{y = f(a) + f'(a)(x-a)}$$

So the tangent at x_n is

$$y = f(x_n) + f'(x_n)(x - x_n)$$

This intersects $y=0$ when

$$0 = f(x_n) + f'(x_n)\,x - f'(x_n)\,x_n$$

$$\frac{-f(x_n) + f'(x_n)\,x_n}{f'(x_n)} = x$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$ → This becomes our new approximation

So

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

In summary:

1) Guess a solution, x_1
2) Use x_1 to find x_2, x_2 to find x_3, and so on, using $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Ex. 1 Use Newton's method to approximate the solution of $x^3 + 3x + 1 = 0$.

Make a guess $x_1 = -0.5$. $f' = 3x^2 + 3$ so $x_2 = x_1 - \dfrac{x_1^3 + 3x_1 + 1}{3x_1^2 + 3} = -0.33$

Now repeat $x_3 = x_2 - \dfrac{x_2^3 + 3x_2 + 1}{3x_2^2 + 3} = -0.322$. You can carry until no difference is given by your calculations ~ -0.3221853546.

We can use dy and dx as estimates for Δy and Δx when small. We call these differentials. The differential dx is an independent variable however dy is dependent and $dy = f'(x)dx$.

Ex. 2 Suppose you measured a radius of $r = 12.2 \pm 0.1$ cm. What effect would the uncertainty have on the surface area of a sphere with this radius?

It would affect the surface area by dS. $S = 4\pi r^2$

$$\begin{aligned} dS &= 8\pi r\,dr \\ &= 8\pi(12.2)(0.1) \\ &\doteq 30.7 \text{ cm}^2 \end{aligned}$$

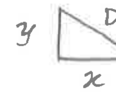
$\therefore$ The surface area could vary by up to 30.7 cm^2. I.e. $S = (4\pi(12.2)^2 \doteq)\ 1870 \pm 30 \text{ cm}^2$.

Homework: pg. 229 # 15, 17–19, 21, 22, 31, 42, 46, 49, 51

4.6 Related Rates

Any equation involving two or more variables that are differentiable functions of time can be differentiated using the chain rule to see how the corresponding rates relate.

E.g. the distance formula $D=\sqrt{x^2+y^2}$

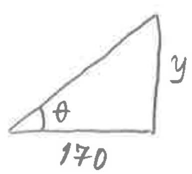


if x and y change with time then $\dfrac{dD}{dt}=\dfrac{1}{2}(x^2+y^2)^{-1/2}\left(2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}\right)$.

Ex. 1 If the radius and height of a cone can be varied with time determine how dV/dt, dr/dt and dh/dt are related.

$$V=\frac{\pi}{3}r^2h \qquad \frac{dV}{dt}=\frac{\pi}{3}\left(2rh\frac{dr}{dt}+r^2\frac{dh}{dt}\right)$$

Ex. 2 A balloon is rising 0.14 rad/min when a range finder 170 m from the balloon's lift-off point measures an angle of $\pi/4$ between the ground and the path to the balloon. How fast is the balloon rising at that instant?



$\theta=\pi/4 \quad \dfrac{d\theta}{dt}=0.14$ rad/min

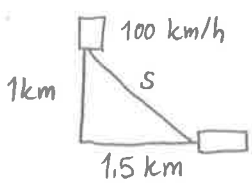
$\dfrac{dy}{dt}=?$

$\tan\theta=\dfrac{y}{170} \Rightarrow y=170\tan\theta$

$$\frac{dy}{dt}=170\sec^2\theta\frac{d\theta}{dt}=170\sec^2\left(\frac{\pi}{4}\right)(0.14)$$
$$=170(2)(0.14)=47.6\ \text{m/min}$$

Strategy: 1) Draw picture 2) List givens 3) List unknown 4) Make equation 5) Take derivative 6) Solve

Ex. 3 A police car travelling 100 km/h 1 km north of an intersection spots a car speeding 1.5 km east of the same intersection. The radar determines the distance between them is increasing by 50 km/h. How fast is the car?



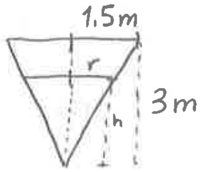
$ds/dt=50$ km/h $\quad dy/dt=-100$ km/h (y is decreasing) $\quad dx/dt=?$

$y=1$ km $\quad x=1.5$ km

$s^2=x^2+y^2 \Rightarrow s=\sqrt{x^2+y^2}$

$$\cancel{2}s\frac{ds}{dt}=\cancel{2}x\frac{dx}{dt}+\cancel{2}y\frac{dy}{dt} \Rightarrow \left(s\frac{ds}{dt}-y\frac{dy}{dt}\right)\frac{1}{x}=\frac{dx}{dt}=\frac{\left(\sqrt{1^2+1.5^2}(50)-(1)(-100)\right)}{(1.5)}$$
$$\doteq 127\ \text{km/h}$$

Ex. 4 Water fills a conical tank with a height of 3 m and radius of 1.5 m at a rate of 0.006 m^3/s. How fast is the water level rising when the water is 2 m deep?



$h = 2m \quad dV/dt = 0.006 \; m^3/s$

$\frac{dh}{dt} = ?$

$V = \frac{1}{3}\pi r^2 h$

$\frac{r}{h} = \frac{1.5}{3}$ (by similar triangles)

$\Rightarrow r = \frac{h}{2}$

$\Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$$V = \frac{\pi}{12} h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} = \frac{4}{\pi (2)^2}(0.006) \doteq 0.002 \text{ m/s } (= 2 \text{ mm/s})$$

Homework: pg. 237 # 1, 4–14, 17, 20, 22, 24*, 28, 29, 31, 33, 34