3.1 Derivative of a Function

Definition: Derivative

the derivative of the function $f(x)$ is the function $f'(x)$ where

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

If $f'(x)$ exists we say that f is differentiable at x. If f is differentiable at every point of its domain we say that f is a differentiable function.

The derivative can also be written as

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Ex. 1 Find the derivative of $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{x^3}+3x^2h+3xh^2+h^3) - \cancel{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2+3x\overset{\to 0}{h}+\overset{\to 0}{h^2})}{\cancel{h}}$$

$$= 3x^2 \qquad \square$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

or use Pascal's triangle

Ex. 2 Find the derivative of $f(x) = \sqrt{x}$

$$f'(a) = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$$

$$= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$$

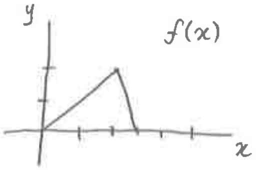
$$= \lim_{x\to a} \frac{\cancel{(x-a)}}{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})}$$

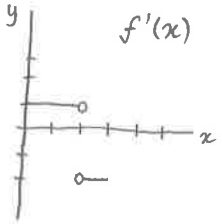
$$= \frac{1}{2\sqrt{a}} \qquad \square$$

Notation

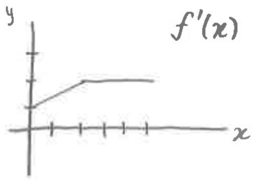
Lagrange	Leibniz	Newton	Euler
y' "y prime"	$\frac{dy}{dx}$ "derivative of y with respect to x"	$\dot{y}$	Dy
$f'(x)$ "f prime (of x)"	$\frac{d\,f(x)}{dx}$	$\dot{f}$	Df
$f''(x)$ "f double prime"	$\frac{d^2f(x)}{dx^2}$ "second derivative of f with respect to x"	$\ddot{f}$	D^2f

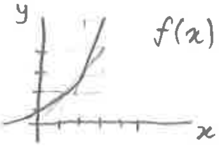
Ex. 3 Draw the graph of $f'(x)$ given the graph of $f(x)$





Ex. 4 Draw a possible graph of $f(x)$ given the graph of $f'(x)$





Note that we can start anywhere. We will learn more about this when we get to integrals.

A function is differentiable on $[a,b]$ if it is differentiable at everypoint in (a,b), and $\lim_{h\to0^+}\frac{f(a+h)-f(a)}{h}$ (the right hand derivative at a) and $\lim_{h\to0^-}\frac{f(b+h)-f(b)}{h}$ (the left hand derivative at b) exist.

For a function to be differentiable at a point, the limit must exist, therefore

$$\lim_{h\to0^+}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0^-}\frac{f(x+h)-f(x)}{h}$$

Ex. 5 Show that $f(x)=\begin{cases}x^2 & x\le0\\ x & x>0\end{cases}$ is not differentiable at $x=0$.

$$\lim_{h\to0^+}\frac{(x+h)-x}{h}=1 \neq \lim_{h\to0^-}\frac{(x+h)^2-x^2}{h}=\lim_{h\to0}\frac{x^2+2x\cancel{h}+h^2-x^2}{h}=\lim_{h\to0}\frac{\cancel{h}(2x+\cancel{h}^{0})}{\cancel{h}}=2x=0 \text{ @ } x=0$$

∴ The limit DNE & $f(x)$ is not differentiable at $x=0$.

Homework: pg. 101 # 1, 3-12, 16-18, 21-23, 25, 28*

3.2 Differentiability

Recall that the limit must exist for a function to be differentiable. If the limit does not exist, then the function isn't differentiable. Here are some cases:

a corner

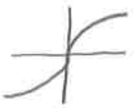
e.g. $f(x)=|x|$

a cusp

e.g. $f(x)=x^{2/3}$

a vertical tangent

e.g. $f(x)=\sqrt[3]{x}$



a discontinuity

e.g. $u(x)=\begin{cases}-1 & x<0\\ 1 & x\geq 0\end{cases}$

"Unit step function"

Ex. 1 Where is $f(x)=|x-2|+3$ not differentiable?

$g(x)=|x| \Rightarrow g(x-2)+3=f(x)$ $g(x)$ is not differentiable at $(0,0)$ $[x=0]$ so $f(x)$ is not differentiable at $(0+2,0+3)=(2,3)$ $[x=2]$.

A differentiable function appears smooth and linear if we "zoom in".

Theorem: Differentiability implies Continuity

$f'(a)$ exists $\Rightarrow$ f is continuous at a.

proof: We need to show that $\lim_{x\to a} f(x)=f(a)$ or $\lim_{x\to a} f(x)-f(a)=0$

$$\lim_{x\to a} f(x)-f(a) = \lim_{x\to a} (x-a)\frac{f(x)-f(a)}{(x-a)} \qquad [x-a\neq 0]$$

$$= \lim_{x\to a}(x-a)\cdot\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$= 0\cdot f'(a)$$

$$=0 \qquad \square$$

Homework: pg. 111 # 1-3, 7-10, 13, 14, 16, 18, 19, 29

Theorem: Intermediate Value Theorem for Derivatives

a & b are any two points in an interval where f is differentiable

$\Rightarrow$ f' takes on every value between $f'(a)$ and $f'(b)$

Ex. 2 Does any function have the unit step function, $u(x)$, as its derivative?

$[(A\Rightarrow B)\Leftrightarrow(\neg B\Rightarrow\neg A)]$ Assume $\exists f: f'=u$ pick any $a<0$ and $b>0$. IVT $\Rightarrow$ $f'=u$ takes on every value between $f'(a)=u(a)=-1$ and $f'(b)=u(b)=1$. Contradiction: u does not take on any value between -1 and 1. $\therefore$ Assumption is false. $\therefore$ $\nexists f: f'=u$. $\square$ [or $u=f'$ does not take on any value between $u(a)=-1$ and $u(b)=1$ $(a<0, b\geq 0)$ $\Rightarrow$ $\nexists\, a \vee b$ where f is differentiable. $\therefore$ a and b cover all $\mathbb{R}$, f is not differentiable. $\therefore$ f' DNE so $f'\neq u$. $\square$]

3.3 Rules for Differentiation

Derivative of a Constant Function

$$c \in \mathbb{R} \qquad \frac{d(c)}{dx} = 0$$

Proof: $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\frac{c-c}{h} = \lim\limits_{h\to 0} 0 = 0 \quad \square$

Power Rule

$$n \in \mathbb{Z}^+ \qquad \frac{d(x^n)}{dx} = nx^{n-1}$$

Proof: $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\frac{(x+h)^n-(x)^n}{h} = \lim\limits_{h\to 0}\frac{(\cancel{x^n}+nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\cdots+nxh^{n-1}+h^n)-\cancel{x^n}}{h}$

$$= \lim_{h\to 0}\frac{\cancel{h}\left(nx^{n-1}+\binom{n}{2}\overset{\to 0}{\cancel{x^{n-2}h}}+\cdots+\overset{\to 0}{\cancel{nxh^{n-2}}}+\overset{\to 0}{\cancel{h^{n-1}}}\right)}{\cancel{h}}$$

$$= nx^{n-1} \quad \square$$

or $\lim\limits_{h\to 0}\frac{(x+h)^n-x^n}{h} = \lim\limits_{h\to 0}\frac{(x+h-x)((x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1})}{h}$

$\because \; a^n-b^n = (a-b)(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1})$ with $a=x+h$ and $b=x$

$$= \lim_{h\to 0}\frac{h}{h}((x+\overset{0}{\cancel{h}})^{n-1}+x(x+\overset{0}{\cancel{h}})^{n-2}+\cdots+x^{n-2}(x+\overset{0}{\cancel{h}})+x^{n-1}) = nx^{n-1} \quad \square$$

Constant Multiple Rule

$c \in \mathbb{R}$ u differentiable function $\qquad \frac{d(cu)}{dx} = c\frac{du}{dx}$

Proof: $\lim\limits_{h\to 0}\frac{cu(x+h)-cu(x)}{h} = \lim\limits_{h\to 0} c\,\frac{u(x+h)-u(x)}{h} = c\lim\limits_{h\to 0}\frac{u(x+h)-u(x)}{h} = c\frac{du}{dx} \quad \square$

Sum and Difference Rule

where u and v are differentiable functions: $\qquad \frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

Proof: $\lim\limits_{h\to 0}\frac{(u\pm v)(x+h)-(u\pm v)(x)}{h} = \lim\limits_{h\to 0}\frac{(u(x+h)\pm v(x+h))-(u(x)\pm v(x))}{h}$

$$= \lim_{h\to 0}\frac{u(x+h)-u(x)\pm v(x+h)\mp v(x)}{h} = \lim_{h\to 0}\left(\frac{u(x+h)-u(x)}{h}\right) \pm \left(\frac{v(x+h)-v(x)}{h}\right)$$

$$= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \pm \lim_{h\to 0}\frac{v(x+h)-v(x)}{h} = \frac{du}{dx} \pm \frac{dv}{dx} \quad \square$$

Ex. 1 Differentiate $f(x) = x^3 - 2x^2 + \frac{3}{2}x - 5$

$$\frac{df}{dx} = \frac{d(x^3)}{dx} - 2\frac{d(x^2)}{dx} + \frac{3}{2}\frac{d(x)}{dx} - \frac{d(5)}{dx}$$ (sum & difference + constant multiple rules)

$$= 3x^2 - 2(2x) + \frac{3}{2}(x^0) - 0$$ (product + constant function rules)

$$= 3x^2 - 4x + \frac{3}{2}$$

Ex. 2 Find any horizontal tangents of $f(x) = x^4 - 2x^2 + 2$

$$f'(x) = 4x^3 - 4x$$
$$0 = 4x(x^2 - 1)$$
$$\Rightarrow x = 0, \quad x = 1, \quad \text{or} \quad x = -1$$
$$f(0) = 2 \quad f(1) = 1 \quad f(-1) = 1$$

$\therefore$ horizontal tangents are located at (0,2) (1,1) and (-1,1).

Product Rule

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$ where u & v are differentiable functions

$[f = uv \Rightarrow f' = u'v + v'u]$

Proof: $$\lim_{h\to 0} \frac{uv(x+h) - uv(x)}{h} = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$
$$= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$
$$= \lim_{h\to 0} u(x+h)\left(\frac{v(x+h) - v(x)}{h}\right) + \lim_{h\to 0} v(x)\left(\frac{u(x+h) - u(x)}{h}\right)$$
$$= u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx} \quad \square$$

Ex. 3 Differentiate $f(x) = (x^2+1)(x^3+2)$

$$f'(x) = (x^2+1)(3x^2) + (x^3+2)(2x)$$
$$= 3x^4 + 3x^2 + 2x^4 + 4x$$
$$= 5x^4 + 3x^2 + 4x$$

Ex. 4 $f = uv$ find $f'(2)$ if $u(2) = 3$ $u'(2) = -4$ $v(2) = 1$ and $v'(2) = 2$

$$f' = uv' + vu' \quad \Rightarrow \quad f'(2) = u(2)v'(2) + v(2)u'(2)$$
$$= 3(2) + (1)(-4)$$
$$= 6 - 4 = 2$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ where u & v are differentiable functions and $v \neq 0$

$$\left[f = \frac{u}{v} \Rightarrow f' = \frac{u'v - v'u}{v^2}\right]$$

Proof: $$\lim_{h\to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} = \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{h\,v(x+h)v(x)}$$
$$= \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{h(v(x+h)v(x))}$$

$$= \lim_{h\to 0} \frac{v(x)\,\underline{(u(x+h)-u(x))}\!\!\!\!\!\!\!\!\!\!\!\!\;\frac{}{h} }{} $$

$$= \lim_{h\to 0} \frac{v(x)\,\frac{(u(x+h)-u(x))}{h} + u(x)\,\frac{(v(x)-v(x+h))}{h}}{v(x+h)\,v(x)} = \frac{v(x)\,\frac{d(u(x))}{dx} - u(x)\frac{d(v(x))}{dx}}{(v(x))^2} \qquad \square$$

Would the power rule still apply for negative integer powers?

Say $n \in \mathbb{Z}^+$ and $f(x) = x^{-n}$

$$\frac{d(x^{-n})}{dx} = \frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{\frac{d(1)}{dx}\cdot x^n - \frac{d(x^n)}{dx}(1)}{(x^n)^2} = \frac{0 - nx^{n-1}}{x^{2n}} = -n\,x^{n-1-2n}$$

$= -n\,x^{-n-1} = (-n)\,x^{(-n)-1}$ which is the same result as the power rule used directly on $x^{(-n)}$. Therefore the power rule can be used with negative integer powers as well provided that $x \neq 0$.

Ex. 5 Find the equation of the tangent line to the curve $f(x) = \frac{x^2+3}{2x}$ at the point $(1,2)$.

$$f'(x) = \frac{(2x)(2x)-(2)(x^2+3)}{(2x)^2} = \frac{4x^2-2x^2-6}{4x^2} \quad \text{or} \quad (2x)\left(\tfrac{1}{2}x^{-1}\right) + \left(-\tfrac{1}{2}x^{-2}\right)(x^2+3)$$

$$f'(1) = \frac{4-2-6}{4} = -1 \qquad \text{or } f'(1) = (2)(\tfrac{1}{2}) - \tfrac{1}{2}(1+3)$$

$$= 1-2 = -1$$

$y = -x + b$

$2 = -(1) + b$

$b = 3$ $\quad\therefore\ y = -x+3$ is the tangent line at $(1,2)$.

Ex. 6 Find the first four derivatives of $f(x) = x^3 - 5x^2 + 2$

$y' = 3x^2 - 10x \qquad y''' = 6$

$y'' = 6x - 10 \qquad y^{(4)} = 0$

Homework: pg. 120 # 2, 3, 5, 7, 9-13, 15, 16, 18, 19, 21, 24, 26-29, 32, 33, 36-39

3.4 Instantaneous Rates of Change

Recall that the instantaneous rate of change of a function $f(x)$ at $x=a$ is just $f'(a)$.

Ex.1 Find the rate of change of the area of a circle in general and when $r=3$.

$A=\pi r^2 \quad A'=2\pi r \qquad A'(3)=2\pi(3)=6\pi \doteq 18.8$

Definition: Instantaneous Velocity

The derivative of the position as a function of time at a certain time is the instantaneous velocity. $\quad v(t)=s'(t)$ where $s(t)$ is the position function

Definition: Speed

Speed $=|v(t)|=\left|\frac{ds}{dt}\right|$

Definition: Acceleration

The derivative of the velocity is the acceleration.

$$a(t)=v'(t)=s''(t)$$
$$=\frac{dv}{dt}=\frac{d^2s}{dt^2}$$

The acceleration due to gravity is $9.8\ m/s^2$ or $32\ ft/sec^2$.

Ex. 2 The height in feet of a rock launched into the air after t seconds is modelled by the equation $s(t)=160t-16t^2$.

(a) How high does the rock go?

at its highest point the rock has zero velocity.

$\frac{ds}{dt}=160-32t=0 \Rightarrow t=\frac{160}{32}=5\ s$

(b) What is the velocity when it is 256 ft in the air?

$t=?$ $\quad 256=160t-16t^2 \Rightarrow 16t^2-160t+256=0 \Rightarrow 16(t^2-10t+16)=0 \Rightarrow 16(t-8)(t-2)=0$

$\therefore t=2$ or $t=8$

$v(2)=s'(2)=160-32(2)=96$ ft/sec $\qquad v(8)=s'(8)=160-32(8)=-96$ ft/sec

(c) What is the rock's acceleration?

$a(t)=v'(t)=-32$ ft/sec^2

(d) When does the rock hit the ground?

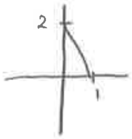
$s=0 \Rightarrow 0=160t-16t^2 \Rightarrow 0=16t(10-t) \Rightarrow t=10$ sec

Ex. 3 Gregor Mendel showed that the proportion of smooth-skinned peas in the next generation, y is related to the relative frequency of the gene for smooth skin in peas, p, by the relation

$$y = 2p - p^2$$

where is the function most sensative to changes?

$$y' = 2 - 2p$$



The slope is greatest when p is close to zero, so that is where it is the most sensative. That means that adding a few more dominant genes into a highly recessive population has more of an effect on later generations than adding the same amount to a dominant population.

Definition: Marginal Cost

The marginal cost is the rate of change of cost, c, with respect to the level of production, x, so it is $\frac{dc}{dx} = c'(x)$. It is loosely defined sometimes as the extra cost of producing one more unit.

Ex. 4 If it costs $c(x) = x^3 - 6x^2 + 15x$ to produce x radiators and 10 radiators are produced per day, find the marginal cost.

$c'(x) = 3x^2 - 12x + 15$ $\quad$ $c'(10) = 3(100) - 12(10) + 15 = 300 - 120 + 15 = 195$ dollars per extra unit a day.

Homework: pg. 129 # 2-5, 9, 10, 12, 13, 20, 24, 25, 28, 30, 32, 34, 37*, 38*

3.5 Derivatives of Trigonometric Functions

The derivative of $y = \sin x$ is $y' = \cos x$:

$$\frac{d}{dx} \sin x = \cos x$$

Proof: $$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \sin x \underset{\substack{\uparrow \\ 0}}{\left(\frac{\cos h - 1}{h}\right)} + \cos x \underset{\substack{\uparrow \\ 1}}{\left(\frac{\sin h}{h}\right)} = \sin x\,(0) + \cos x\,(1) = \cos x \qquad \square$$

Aside: $$\lim_{h\to 0} \frac{\cos h - 1}{h} \frac{(\cos h + 1)}{(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h\to 0} \frac{\sin^2 h}{h} \frac{1}{\cos h + 1}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1} = 1\left(\frac{0}{1+1}\right) = 0$$

The derivative of $y = \cos x$ is $y' = -\sin x$:

$$\frac{d}{dx} \cos x = -\sin x$$

We can write a simple proof for this once we learn the chain rule.

Ex. 1 Find the derivatives of $f(x) = x^2 \sin x$ and $g(x) = \dfrac{\cos x}{1 - \sin x}$.

$f'(x) = 2x \sin x + x^2 \cos x$ $\qquad$ $$g'(x) = \frac{-\sin x(1 - \sin x) - (-\cos x)(\cos x)}{(1 - \sin x)^2}$$

$$= (-\sin x + \sin^2 x + \cos^2 x)/(1 - \sin x)^2$$

$$= (1 - \sin x)/(1 - \sin x)^2$$

$$= \frac{1}{1 - \sin x}$$

Ex. 2 The distance of a mass hanging on a spring from its equilibrium position after being stretched 5 units and released can be described by the equation

$$s(t) = 5 \cos t$$

where t is the time after being released. Determine the weight's velocity and acceleration at time t.

$s'(t) = -5 \sin t$ $\qquad$ $s''(t) = -5 \cos t$

Note that $|s(t)|$ is maximum when $t = k\pi$. This is also where $v = s'$ is zero. The acceleration is zero when $t = (2k+1)\pi/2$. This is also where $s(t) = 0$, so the equilibrium position. Here $F_K = F_G$ so $a = 0$ by Newton's Second Law.

Since we know $\tan x = \sin x / \cos x$, $\cot x = \cos x / \sin x$, $\sec x = 1/\cos x$, and $\csc x = 1/\sin x$ we can also find the derivatives of these functions as well:

$\frac{d}{dx} \tan x = \sec^2 x$ $\quad$ $\frac{d}{dx} \cot x = -\csc^2 x$ $\quad$ $\frac{d}{dx} \sec x = \sec x \tan x$ $\quad$ $\frac{d}{dx} \csc x = -\csc x \cot x$

Ex. 3 Find the equation for the tangent and normal lines to $f(x) = \frac{\tan x}{x}$ at $x = 2$.

$$f'(x) = \frac{(\sec^2 x)(x) - (1)(\tan x)}{x^2} \Rightarrow f'(2) = \frac{\frac{2}{\cos^2(2)} - \tan(2)}{(2)^2} \doteq 3.4$$

$m = 3.4 \quad x = 2 \quad f(2) = -1.1$

$y - mx = b$

$-1.1 - (3.4)(2) = b$

$b = -7.9$

$y = 3.4x - 7.9$

$m = -0.29 \quad x = 2 \quad f(2) = -1.1$

$b = y - mx$

$= -1.1 - (-0.29)(2)$

$= -0.52$

$y = -0.29x - 0.52$

Ex. 4 Find y'' if $y = \sec x$.

$y' = \sec x \tan x \qquad y'' = (\sec x \tan x)(\tan x) + (\sec^2 x)\sec x$

$= \sec^3 x + \sec x \tan^2 x$

Homework: pg. 140 # 1-12, 15, 17, 20, 26, 28, 30, 33, 36

mention # 25

3.6 Chain Rule

Ex.1 Find the derivative of $y=(2x^2+1)^2$

$y=4x^4+4x^2+1 \qquad y'=16x^3+8x = 8x(2x^2+1) = 2(4x)(2x^2+1)$

Notice how $\frac{d(2x^2+1)}{dx}=4x$ and $\frac{du^2}{du}=2u$. If $u=(2x^2+1)$, then $u'=4x$ and $y=u^2$. y' is then $y'=2(2x^2+1)(4x)=2u\cdot u'$. This is the chain rule.

The Chain Rule:

$$(f\circ g)(x)=f(g(x)) \Rightarrow (f\circ g)'(x)=f'(g(x))\cdot g'(x)$$

$$y=f(u) \quad u=g(x) \Rightarrow \quad y'=f'(u)\,g'(x)$$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

Proof: $\lim\limits_{h\to 0}\frac{(f\circ g)(x+h)-(f\circ g)(x)}{h}=\lim\limits_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\;\frac{g(x+h)-g(x)}{g(x+h)-g(x)}$

$=\lim\limits_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\;\frac{g(x+h)-g(x)}{h}=\lim\limits_{b\to a}\frac{f(b)-f(a)}{b-a}\cdot\lim\limits_{h\to 0}\frac{g(x+h)-g(x)}{h}$

[let $b=g(x+h)$ and $a=g(x)$. as $h\to 0$, $b\to a$.]

$=f'(a)\cdot g'(x)=f'(g(x))\cdot g'(x) \quad \square$

Ex. 2 Find the velocity of the position function $x(t)=\cos(t^2+1)$.

$u=t^2+1 \Rightarrow x=\cos u$ $\qquad \frac{dx}{dt}=\frac{dx}{du}\frac{du}{dt}=-\sin u\cdot 2t=-2t\sin(t^2+1)$.

$\frac{du}{dt}=2t \qquad \frac{dx}{du}=-\sin u$

Ex. 3 Find the derivative of $\sin(x^2+x)$

$$y'=\underbrace{\cos(x^2+x)}_{\text{outside}}\cdot\underbrace{(2x+1)}_{\text{inside}}$$

Ex. 4 Differentiate $g(x)=\tan(5-\sin 2x)$

$g'(x)=\sec^2(5-\sin 2x)\cdot(0-(\cos 2x)\cdot(2))$

$\quad=-(2\cos 2x)(\sec^2(5-\sin 2x))$

Ex. 5 Find the slope of the tangent to $y=\sin^5 x$ when $x=\pi/3$

$y=(\sin x)^5 \qquad y'=5(\sin x)^4(\cos x)=5\sin^4 x\cos x$.

$y'(\pi/3)=5(\sin\pi/3)^4(\cos\pi/3)=5\left(\sqrt{3}/2\right)^4\left(1/2\right)=\frac{5(9)}{2(16)}=\frac{45}{32}$

Ex. 6 Show that the slope of $y = 1/(1-2x)^3$ is everywhere positive.

$$y = (1-2x)^{-3} \qquad y' = -3(1-2x)^{-4}(-2) = \frac{6}{(1-2x)^4}$$

$(1-2x)^4$ is always positive, $\therefore$ y' is always positive as well.

Ex. 7 Prove $\frac{d\cos x}{dx} = -\sin x$

$$\cos x = \sin\left(\tfrac{\pi}{2}-x\right) \Rightarrow \frac{d\cos x}{dx} = \frac{d}{dx}\sin\left(\tfrac{\pi}{2}-x\right) = \underbrace{\cos\left(\tfrac{\pi}{2}-x\right)}_{\sin x}(-1) = -\sin x \quad \square$$

Ex. 8 Show that the derivative of $\sin x$ in degrees is not just $\cos x$.

$$\frac{d}{dx}\sin(x^\circ) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x^\circ)$$

This term would compound with repeated differentiation.

Homework: pg. 146 # 1-7, 9, 12-14, 16-18, 20, 22-24, 32-34, 37, 40, 52, 57 b-e, 58, 66*

3.7 Implicit Differentiation

Ex. 1 Find $\frac{dy}{dx}$ if $y^2 = x$

Use the chain rule on the left $2y \cdot y' = 1$ or $2y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$

Notice how we used the chain rule on the left side to differentiate y^2. This is known as implicit differentiation. It is useful when we cannot isolate for y.

Ex. 2 Find the slope of the circle $x^2+y^2=25$ at the point $(3, 4)$.

$$2x + 2y \cdot y' = 0 \quad \Rightarrow \quad y' = \frac{-2x}{2y} = \frac{-2(3)}{2(4)} = -\frac{3}{4}$$

Ex. 3 Show that dy/dx is defined for all $x \in \mathbb{R}$ if $2y = x^2 + \sin y$

$$2y' = 2x + \cos y \cdot y' \quad \Rightarrow \quad 2y' - y'\cos y = 2x$$

$$y'(2-\cos y) = 2x$$

$$y' = \frac{2x}{2-\cos y}$$

↗ only zero when $\cos y = 2$ which is not possible

Ex. 4 Find the tangent and normal to the ellipse $x^2 - xy + y^2 = 7$ at $(-1, 2)$

$$2x - y - y'x + 2yy' = 0 \quad \Rightarrow \quad y'(-x+2y) = y - 2x$$

$$y' = \left.\frac{y-2x}{2y-x}\right|_{(-1,2)}$$

$$= \frac{(2)-2(-1)}{2(2)-(-1)}$$

$$= 4/5$$

tangent	normal
$b = y - mx$	$b = y - mx$
$= 2 - \frac{4}{5}(-1)$	$b = 2 + \frac{5}{4}(-1)$
$= \frac{14}{5}$	$= 3/4$
$y = \frac{4}{5}x + \frac{14}{5}$	$y = -\frac{5}{4}x + \frac{3}{4}$

Ex. 5 Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$

$$6x^2 - 6y \cdot y' = 0 \quad \Rightarrow \quad y' = \frac{-6x^2}{-6y} = \frac{x^2}{y} \qquad (y \neq 0)$$

$$\Rightarrow \quad y'' = \frac{2x \cdot y - y'x^2}{y^2}$$

$$= \frac{2xy - \left(\frac{x^2}{y}\right)x^2}{y^2} = \frac{2x}{y} - \frac{x^4}{y^3} \qquad (y \neq 0)$$

We have been using the power rule for rational numbers, e.g. $y=\sqrt{x}=x^{1/2}$, however we haven't proved that it works for rational numbers, only integers.

Ex. 6 Prove the power rule for rational powers

Suppose $y=\sqrt[q]{x^p}=x^{p/q}$ $(q>0)$ $\Rightarrow$ $y^q=x^p$ $\Rightarrow$ $qy^{q-1}y'=px^{p-1}$

$$y'=\frac{px^{p-1}}{qy^{q-1}} \quad (y\neq 0)$$

$$=\frac{p}{q}\frac{x^{p-1}}{(x^{p/q})^{q-1}}=\frac{p}{q}\frac{x^{p-1}}{x^{p-p/q}}$$

$$=\frac{p}{q}x^{p-1-(p-p/q)}=\frac{p}{q}x^{p/q-1} \ \square$$

Ex. 7 Differentiate $(\cos x)^{-1/5}$

$$-\frac{1}{5}(\cos x)^{-\frac{1}{5}-1}(-\sin x)=\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$$

Homework: pg. 155 # 2, 3, 6-10, 12, 13, 17, 20, 22-24, 26, 27, 34, 38, 41, 43, 48

3.8 & 3.9 Derivatives of Special Functions

Theorem:

f is differentiable at every point on I and df/dx is never zero on I $\Rightarrow$ f has an inverse, and f^{-1} is differentiable at every point on $f(I)$.

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \quad |x|<1 \qquad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \quad |x|<1$$

Ex.1 Prove that $d/dx \sin^{-1}x = 1/\sqrt{1-x^2}$

$$y = \sin^{-1}x \Rightarrow \quad \sin y = x \quad \Rightarrow \quad \cos y \cdot y' = 1 \quad \Rightarrow y' = \frac{1}{\cos y}$$

$$y' = \frac{1}{\cos(\sin^{-1}x)} \qquad \cos^2 x + \sin^2 x = 1 \quad \Rightarrow \quad \cos x = \sqrt{1-\sin^2 x}$$

$$= \frac{1}{\sqrt{1-[\sin(\sin^{-1}x)]^2}}$$

$$= \frac{1}{\sqrt{1-x^2}} \qquad \square$$

You may have already seen the letter "e" before in math. It is Euler's number:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

another property of this number is: $\lim_{h\to 0}\frac{e^h-1}{h} = 1$.

Using this we can find the derivative of $f(x) = e^x$:

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} e^x\left[\frac{e^h-1}{h}\right] = (e^x)(1) = e^x$$

Thus $\quad \dfrac{de^x}{dx} = e^x$

Ex. 2 The number of students infected by the flu after t days is modelled by $P(t) = \dfrac{100}{1+e^{3-t}}$. How fast is the flu spreading after 3 days? How many were initially infected?

$$P(0) = \frac{100}{1+e^3} = 4.7 \doteq 5 \text{ students}$$

$$P(t) = 100(1+e^{3-t})^{-1}$$

$$P'(t) = -100(1+e^{3-t})^{-2}(e^{3-t}(-1))$$

$$= \frac{e^{3-t}\,100}{(1+e^{3-t})^2}$$

$$P'(3) = \frac{e^0\,100}{(1+e^0)^2} = \frac{100}{4} = 25 \text{ students per day}$$

$\frac{d}{dx} a^x = a^x \ln a$ $(a>0, a\neq 1)$ proof: $\frac{d}{dx} a^x = \frac{d\, e^{\ln(a^x)}}{dx} = \frac{d\, e^{x\ln a}}{dx} = e^{x\ln a}\cdot \ln a = a^x \cdot \ln a$ □

Note that $\ln x = \log_e x = \frac{\log x}{\log e}$

Ex. 3 Where does $y = 2^t - 3$ have a slope of 21?

$y' = 2^t \ln 2$

$2^t \ln 2 = 21$

$2^t = \frac{21}{\ln 2}$

$t \ln 2 = \ln\left(\frac{21}{\ln 2}\right)$

$t = \frac{\ln(21) - \ln(\ln(2))}{\ln 2}$

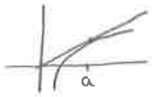
$\doteq 4.9$

$y \doteq 2^{4.9} - 3 \doteq 27.3$ $\therefore$ @ $(4.9, 27.3)$

$\frac{d \ln x}{dx} = \frac{1}{x}$ proof: $y = \ln x \Rightarrow e^y = x \Rightarrow e^y \cdot y' = 1 \Rightarrow y' = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$ □

Ex. 4 Find the line that passes through the origin and is tangent to $y = \ln x$.

$m = y' = \frac{1}{x}$



call this point a $\Rightarrow m = \frac{1}{a}$ $y = \ln a$

$y = mx + b \Rightarrow 0 = \frac{1}{a}(0) + b \Rightarrow b = 0$

$\ln a = \frac{1}{a} x \Rightarrow x = a \ln a$ but $x = a$

$\therefore a \ln a = a \Rightarrow \ln a = 1 \Rightarrow a = e$

$\therefore y = \frac{x}{e}$

$\frac{d \log_a x}{dx} = \frac{1}{x \ln a}$ proof: $\frac{d \log_a x}{dx} = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x \ln a}$ □

Ex. 5 Find $\frac{d}{dx} \log_a a^{\sin x}$.

$\frac{a^{\sin x}\cdot \ln a}{a^{\sin x} \ln a} \cdot \cos x = \cos x$

Ex. 6 Find the domain of $\frac{d}{dx} \ln(x-3)$

$y' = \frac{1}{x-3}$ $x \neq 3$ but y is defined only for $(3, \infty)$, so so is y'.

Ex. 7 Find y' for the following:

(a) $y = x^{\sqrt{2}}$

$y' = \sqrt{2}\, x^{\sqrt{2}-1}$

(b) $y = x^x$

$\ln y = x \ln x$

$\frac{y'}{y} = \ln x + \frac{x}{x}$

$y' = y(\ln x + 1) = x^x(\ln x + 1)$

Homework: pg. 170 # 2, 4, 8, 9, 14, 16, 19, 21, 22, 29, 30, 34, 36, 38, 39, 41-43, 47, 49, 53*