2.1 Rates of Change & Limits

Ex. 1 If an object falls $y = 16t^2$ feet in t seconds, determine its average speed after 2s.

$$\bar{v} = \frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2-0} = 32 \text{ ft/s}$$

Ex. 2 Create an equation for the average speed in example 1 for time t after 2s.

$$t_i = 2 \quad t_f = 2+h \Rightarrow \quad \bar{v} = \frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{(2+h)-2} = \frac{16(4+4h+h^2) - 64}{h} = 64 + 16h$$

Definition of Limit: $\left[c, L \in \mathbb{R} \quad \forall \varepsilon \, \exists \delta : \quad 0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon \right]$

$$\Rightarrow$$

$$\lim_{x \to c} f(x) = L$$

"The function f has limit L as x approaches c."

Theorem 1: Properties of Limits

$$L, M, c, k \in \mathbb{R} \quad \lim_{x \to c} f(x) = L \quad \lim_{x \to c} g(x) = M \qquad \lim_{x \to c} k = k$$

$$\Rightarrow \lim_{x \to c} (f(x) + g(x)) = L + M \quad \lim_{x \to c} f(x) \cdot g(x) = L \cdot M \qquad \lim_{x \to c} x = c$$

$$\lim_{x \to c} (f(x) - g(x)) = L - M \quad \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \quad M \neq 0$$

$$\lim_{x \to c} k f(x) = kL \qquad r, s \in \mathbb{Z} \quad s \neq 0 \quad \lim_{x \to c} (f(x))^{r/s} = L^{r/s} \; [\Leftrightarrow \; L^{r/s} \in \mathbb{R}]$$

Ex. 3 Use the above properties to find $\lim_{x \to c} (x^3 + 2x^2 - 5)$

Theorem 2: Polynomial and Rational Functions

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \qquad c \in \mathbb{R} \qquad g(x) = b_n x^n + \cdots + b_0 \quad g(c) \neq 0$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0$$

$$\Rightarrow \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$$

Ex. 4 Evaluate (a) $\lim_{x \to 2} \frac{x^2 - 2x + 5}{x - 3}$ (b) $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1$

Ex. 5 See graphically that $\lim_{x \to 2} \frac{x^2 - 1}{x - 2}$ DNE.

Theorem 3: Two sided Limits

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \;\wedge\; \lim_{x \to c^-} f(x) = L$$

Ex. 6 Determine $\lim_{x \to 1 \,\&\, 2} f(x)$ for $f(x) = \begin{cases} -x+1 & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \\ x - 1 & 2 \leq x \leq 3 \end{cases}$

Theorem 4: The Sandwich Theorem

$$g(x) \le f(x) \le h(x) \quad \forall\, x \ne c \text{ in some interval about } c \qquad \lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

$$\Rightarrow \quad \lim_{x\to c} f(x) = L$$

Ex. 7 Show that $\lim_{x\to 0} [x^2 \sin(1/x)] = 0$

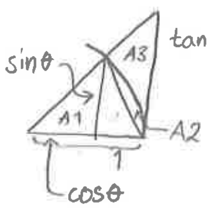
$|\sin x| \le 1$ $\qquad |x^2 \sin\frac{1}{x}| = |x^2|\,|\sin\frac{1}{x}| \le |x^2|\,|1| = x^2$

$$\Rightarrow \qquad -x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$\lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0}(x^2)$ $\quad\therefore$ By the Sandwich Theorem $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$ $\quad\square$

Homework: pg. 62 # 1, 3, 6, 7, 15, 19, 21, 24-26, 45, 46, 56, 63 (Challenge)

If time allows (#63): Prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$



$\tan\theta$ $\left[\because \frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta} \text{ by similar triangles}\right]$

$A1 = \frac{1}{2}\sin\theta \qquad A2 = \frac{\theta}{2\pi}\pi r^2 = \frac{\theta}{2} \qquad A3 = \frac{1}{2}\tan\theta$

$0 < \theta < \frac{\pi}{2}$

$$\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \qquad \lim_{\theta\to 0^+}\cos\theta = 1 \qquad \lim_{\theta\to 0^+} 1 = 1$$

$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$ by Sandwich Theorem

$$f(\theta) = \frac{\sin\theta}{\theta} \qquad f(-\theta) = \frac{\sin(-\theta)}{-\theta} = \frac{-\sin(\theta)}{-\theta} = \frac{\sin\theta}{\theta} = f(\theta)$$

$\therefore f(\theta)$ is even

$$\therefore \lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

$$\therefore \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad \square$$

2.2 Limits Involving Infinity

$$\lim_{x\to\infty}\frac{1}{x}=0 \quad \lim_{x\to-\infty}\frac{1}{x}=0 \quad \lim_{x\to0^+}\frac{1}{x}=\infty \quad \lim_{x\to0^-}\frac{1}{x}=-\infty$$

The properties of limits as $x\to c$ we have already seen also hold when $x\to\pm\infty$.
The Sandwich (aka Squeeze) Theorem also holds for limits as $x\to\pm\infty$.

Definition: Horizontal Asymptote

$y=b$ is the horizontal asymptote of $y=f(x)$ $\Leftrightarrow$ $\lim_{x\to\infty}f(x)=b \lor \lim_{x\to-\infty}f(x)=b$

Definition: Vertical Asymptote

$x=a$ is a vertical asymptote of $y=f(x)$ $\Leftrightarrow$ $\lim_{x\to a^+}f(x)=\pm\infty \lor \lim_{x\to a^-}f(x)=\pm\infty$

Ex. 1 Determine any horizontal asymptotes for $f(x)=\dfrac{2x^2}{x^2+1}$

$$\lim_{x\to\infty}\frac{2x^2}{x^2+1}=\lim_{x\to\infty}\frac{\frac{2x^2}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}=\lim_{x\to\infty}\frac{2}{1+\frac{1}{x^2}}=\frac{2}{1+0}=2$$

Note this also holds as $x\to-\infty$ $\lim_{x\to-\infty}f(x)=2.$

$\therefore y=2$

Ex. 2 Find $\lim_{x\to\infty}\dfrac{\sin x}{x}$.

To use the Squeeze Theorem we need bounds.

$$-1\le\sin x\le1$$

$$-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$$

$$\lim_{x\to\infty}-\frac{1}{x}=0 \qquad \lim_{x\to\infty}\frac{1}{x}=0$$

$\therefore$ By the Squeeze Theorem $\lim_{x\to\infty}\dfrac{\sin x}{x}=0$

$\dfrac{\sin x}{x}$ is an even function, $\therefore$ we can also conclude $\lim_{x\to-\infty}\dfrac{\sin x}{x}=0$

Ex. 3 Evaluate $\lim_{x\to\infty}\dfrac{7x+\sin x}{x}$.

$$\lim_{x\to\infty}\frac{7x+\sin x}{x}=\lim_{x\to\infty}\frac{7x}{x}+\frac{\sin x}{x}=\lim_{x\to\infty}7+\lim_{x\to\infty}\frac{\sin x}{x}=7+0=7$$

Definition: End Behaviour Model

$g(x)$ is a right end behaviour model for $f(x)$ $\Leftrightarrow$ $\lim_{x\to\infty}\dfrac{f(x)}{g(x)}=1$

$g(x)$ is a left end behaviour model for $f(x)$ $\Leftrightarrow$ $\lim_{x\to-\infty}\dfrac{f(x)}{g(x)}=1$

Ex. 4 Show that $\frac{2}{3}x^3$ models $\dfrac{2x^5+x^4-x^2+1}{3x^2-5x+7}$ in its end behaviour.

$$\lim_{x\to\pm\infty}\frac{2x^5+x^4-x^2+1}{3x^2-5x+7}\div\left(\frac{2}{3}x^3\right)=\lim_{x\to\pm\infty}\frac{3(2x^5+x^4-x^2+1)}{2x^3(3x^2-5x+7)}$$

$$=\lim_{x\to\pm\infty}\frac{3(2x^5+x^4-x^2+1)}{2(3x^5-5x^4+7x^3)}\left(\frac{1/x^5}{1/x^5}\right)=\lim_{x\to\pm\infty}\frac{3(2+\frac{1}{x}-\frac{1}{x^3}+\frac{1}{x^5})}{2(3-\frac{5}{x}+\frac{7}{x^2})}=\frac{3(2+0-0+0)}{2(3-0+0)}$$

$=\dfrac{6}{6}=1$ $\square$ Note this is just $\dfrac{2x^5}{3x^2}=\dfrac{2}{3}x^3$, the quotient of the leading terms.

Ex. 5 Find any horizontal asymptotes for $f(x) = \frac{4x^2+5}{2x^3-1}$

$$\lim_{x\to\pm\infty} \frac{(4x^2+5)(1/x^3)}{(2x^3-1)(1/x^3)} = \lim_{x\to\pm\infty} \frac{\frac{4}{x}+\frac{5}{x^3}}{2-\frac{1}{x^3}} = \frac{0+0}{2-0} = 0 \quad \therefore\ y=0$$

Ex. 6 Find $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$

$$\lim_{x\to\infty} \frac{1}{x} = 0 \text{ and } \sin(0) = 0 \quad \therefore \lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = 0$$

We can also write it as: $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{x\to 0^+} \sin x = 0$

In general: $\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right)$ and $\lim_{x\to -\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right)$

Homework: pg. 71 # 1, 4, 9, 10, 17, 25, 26, 29-32, 35, 36, 46, 58

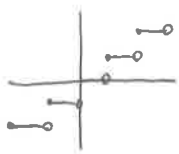
2.3 Continuity

Definition: Continuity at a Point

A function $y = f(x)$ is continuous at an interior point c of its domain $\iff \lim_{x \to c} f(x) = f(c)$

a function $y = f(x)$ is continuous at a left endpoint a or at a right endpoint b of its domain $\iff \lim_{x \to a^+} f(x) = f(a)$ or $\lim_{x \to b^-} f(x) = f(b)$

Ex. 1 Where is the function $f(x) = \lfloor x \rfloor$ (round down and only take the integer part of the number) (also represented as: int x) discontinuous?



$f(x)$ is discontinuous when $x \in \mathbb{Z}$

Types of Discontinuities:

removable   jump   infinite   oscillating

e.g. $\lim_{x \to 0} \sin \frac{1}{x}$ DNE

e.g. $f(x) = \frac{(x+3)(x-2)}{(x-2)}$ has a removable discontinuity at $x = 2$.

A function is continuous on an interval $\iff$ it is continuous at every point on that interval

a function is continuous $\iff$ it is continuous at every point of its domain

Note that $f(x) = \frac{1}{x}$ is a continuous function because $x = 0$ (the discontinuity) is <u>not</u> a part of its domain.

It naturally follows that polynomials are continuous functions.

Theorem: Properties of Continuous Functions

f and g are continuous at $x = c$ $\Rightarrow$ $f+g$, $f-g$, $f \cdot g$, kf $(k \in \mathbb{R})$ & f/g $(g(c) \neq 0)$ are continuous at $x = c$

f is continuous at c & g is continuous at $f(c)$ $\Rightarrow$ $g \circ f$ is continuous at c

Ex. 2 Show that $y = \left| \frac{x \sin x}{x^2+2} \right|$ is continuous.

y is the composition of two continuous functions $f(x) = |x|$ and $g(x) = \frac{x \sin x}{x^2+2}$

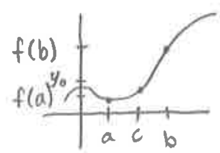
$y = (f \circ g)(x)$. $f(x) = |x|$  is continuous by inspection.

$g(x)$ is the product of two continuous functions x and $\sin x$ divided by the continuous function x^2+2. As this is a quotient we must ensure $x^2+2 \neq 0$ to maintain continuity. This is indeed the case. □

The Intermediate Value Theorem for Continuous Functions:

A function $f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

i.e.
$$y_0 \in [f(a), f(b)] \Rightarrow \exists c \in [a,b] : y_0 = f(c)$$
[f is continuous on $[a,b]$]



Ex. 3 Does any real number equal to 1 less its cube?

i.e. is there an x such that $x = x^3 - 1$?
i.e. is there a solution to $x^3 - x - 1 = 0$?

This is a polynomial and thus a continuous function.
Let $a = 1$ (then $f(a) = f(1) = -1$) and $b = 2$ (then $f(b) = f(2) = 5$).

$$0 \in [f(a), f(b)] = [-1, 5] \wedge f(x) = x^3 - x - 1 \text{ is continuous}$$
$$\Rightarrow$$
$$\exists c \in [a,b] = [1,2] : \quad 0 = f(c) = c^3 - c - 1 \quad \text{by IVT}$$

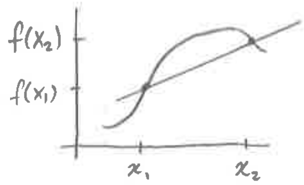
Thus there is a solution (c) to $x^3 - x - 1 = 0$, and moreover it can be found in the interval $[1,2]$. □

Homework: pg. 80 # 2, 7, 12-14, 16, 17, 20, 21, 25, 31, 35-39, 41, 42 (Challenge), 45 (Extra)

2.4 Rates of Change and Tangent Lines

The average rate of change of a function $f(x)$ over an interval Δx is

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



The line connecting $(x_1, f(x_1))$ to $(x_2, f(x_2))$ is known as a secant line. The slope of this line is the average rate of change.

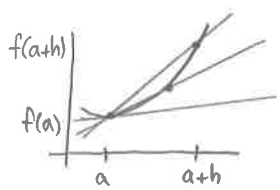
Ex. 1 Determine the average rate of change of $f(x) = x^3 - 2x + 1$ over the interval $[1,3]$

$$\frac{f(3) - f(1)}{3-1} = \frac{22 - 0}{2} = 11$$

Let us re-write our equation for the slope of the secant line. Let $x_1 = a$ and $x_2 = a+h$, then the slope becomes

$$\frac{f(a+h) - f(a)}{(a+h) - (a)} = \frac{f(a+h) - f(a)}{h}$$

} called the "difference quotient of f at a."



Now if h is made smaller and smaller we can see that the slope of the secant line gets closer and closer to the slope of the curve at $f(a)$. The line with a slope equal to the slope of the curve at $f(a)$ and passing through $f(a)$ is known as the tangent line. It is then clear that the slope of the tangent line of f at a would be reached as $h \to 0$.

Definition: Slope of a Curve at a Point

The slope of the curve $y = f(x)$ at the point $(a, f(a))$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists. This can also be written as

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Note that the slope of $f(x)$ at $x = a$ can also be stated as:

- the slope of the tangent to f at a.
- the instantaneous rate of change of f at a.
- the derivative of f at a.

Ex. 2 Find the slope of $f(x) = \frac{1}{x}$ at $x = a$ and at what x value does the slope equal to $-1/4$?

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{\frac{a-(a+h)}{a(a+h)}}{h} = \lim_{h\to 0} \frac{a-a-h}{h(a^2+ah)} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(a^2+ah)}$$

$$= \lim_{h\to 0} \frac{-1}{a^2+a\cancel{h}^{0}} = \frac{-1}{a^2}$$

(The derivative of $f(x) = \frac{1}{x}$ is thus $\frac{-1}{x^2}$.)

The slope will be $-\frac{1}{4}$ when $\frac{-1}{x^2} = \frac{-1}{4} \Rightarrow x = \pm 2$.

Note that the slope is infinite when $x = 0$: ∞ when $x \to 0^+$ & $-\infty$ when $x \to 0^-$.

Ex. 3 Write the equation for the normal to the curve $f(x) = 4 - x^2$ at $x = 1$.
The slope of the normal line will be the negative reciprocal of the slope of the tangent line at that point.

$$m_T = \lim_{h\to 0} \frac{(4-(a+h)^2)-(4-a^2)}{h} = \lim_{h\to 0} \frac{\cancel{4}-(a^2+2ah+h^2)-\cancel{4}+a^2}{h}$$

$$= \lim_{h\to 0} \frac{-a^2-2ah-h^2+a^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(-2a-\cancel{h}^{0})}{\cancel{h}} = -2a$$

$a = 1$ in this case so $m_T = -2$. (We could have subbed in $a = 1$ from the start.)

$$m_N = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + b \qquad x = 1 \quad f(x) = f(1) = 3$$

$$3 = \frac{1}{2}(1) + b \Rightarrow b = 3 - \frac{1}{2} = \frac{5}{2}$$

$\therefore$ $y = \frac{1}{2}x + \frac{5}{2}$ is the equation of the normal line at $x = 1$.

In physics, the slope of the d-t curve is the velocity. Determine the instantaneous velocity of a rock falling according to the relation $y = 4.9t^2$ at $t = 2$ s.

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{4.9(2+h)^2 - 4.9(2)^2}{h} = \lim_{h\to 0} \frac{4.9(\cancel{4}+4h+h^2-\cancel{4})}{h}$$

$$= \lim_{h\to 0} \frac{4.9\cancel{h}(4+\cancel{h}^{0})}{\cancel{h}} = (4.9)(4) \text{ m/s} = 19.6 \text{ m/s}.$$

Homework: pg. 87 # 2-4, 11, 12, 16, 19, 20, 27, 29, 31a (Challenge), 32 (Challenge).